



COLLÈGE  
DE FRANCE  
—1530—



Chaire de Physique Mésoscopique  
Michel Devoret  
Année 2010, 11 mai - 22 juin

**INTRODUCTION AU CALCUL QUANTIQUE**  
***INTRODUCTION TO QUANTUM COMPUTATION***

Troisième Leçon / *Third Lecture*

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10-III-1

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or

<http://www.physinfo.fr/lectures.html>

**PDF FILES OF ALL PAST LECTURES ARE POSTED**

Questions, comments and corrections are welcome!

write to "phymeso@gmail.com"

10-III-2

## CONTENT OF THIS YEAR'S LECTURES

### QUANTUM COMPUTATION FROM THE PERSPECTIVE OF MESOSCOPIC CIRCUITS

1. Introduction, c-bits versus q-bits
2. The Pauli matrices and quantum computation primitives
3. Stabilizer formalism for state representation
4. Clifford calculus
5. Algorithms
6. Error correction

10-III-3

## CALENDAR OF SEMINARS

May 11: Cristian Urbina, (Quantronics group, SPEC-CEA Saclay)

Josephson effect in atomic contacts and carbon nanotubes

May 18: Benoît Douçot (LPTHE / Université Pierre et Marie Curie)

Towards the physical realization of topologically protected qubits

June 1: Takis Kontos (LPA / Ecole Normale Supérieure)

Points quantiques et ferromagnétisme

June 8: Cristiano Ciuti (MPQ, Université Paris - Diderot)

Ultrastrong coupling circuit QED : vacuum degeneracy and quantum phase transitions

June 15: Leo DiCarlo (Yale)

Preparation and measurement of tri-partite entanglement in a superconducting quantum circuit

June 22: Vladimir Manucharian (Yale)

The fluxonium circuit: an electrical dual of the Cooper-pair box?

10-III-4

## **LECTURE III : STABILIZER FORMALISM FOR STATE REPRESENTATION**

1. Motivations
2. Pauli group and stabilizer definition
3. Stabilizer classes for 1 qubit
4. Stabilizer classes for 2 qubits
5. Stabilizer maps

10-III-5

## **OUTLINE**

1. Motivations
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10-III-5a

## MOTIVATION 1: DESCRIBING THE CONTENT OF A QUANTUM REGISTER

Example of content of classical 10 bit register:

0 1 0 1 1 0 1 0 0 1

Example of content of quantum 10 bit register (10 qubits):

$$\begin{aligned}
 |\Psi\rangle = & \frac{1}{2^5} (|0000000000\rangle - |0000000001\rangle + \dots \\
 & \dots + |0101101001\rangle + \dots \\
 & - |1111111101\rangle + |1111111110\rangle - |1111111111\rangle)
 \end{aligned}$$

in general, 1024 terms!

Complexity of description grows exponentially with the number of qubits...

10-III-6a

## MOTIVATION 2: DISTINGUISHING ENTANGLED FROM PRODUCT STATES

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} : \text{PRODUCT STATE} \quad \left( = \frac{(|0\rangle + |1\rangle)_1 (|0\rangle + |1\rangle)_2}{2} \right)$$

$$\frac{|00\rangle + |01\rangle - |10\rangle - |11\rangle}{2} : \text{PRODUCT STATE} \quad \left( = \frac{(|0\rangle - |1\rangle)_1 (|0\rangle + |1\rangle)_2}{2} \right)$$

$$\frac{|00\rangle + |01\rangle - |10\rangle + |11\rangle}{2} : \text{ENTANGLED STATE}$$

10-III-7a

### MOTIVATION 3: IMPROVE SYMMETRY OF BASES

BASIS NAME	KET SET
Computational basis	$\{ 00\rangle,  01\rangle,  10\rangle,  11\rangle\}$
Bell basis	$\left\{ \frac{ 00\rangle +  11\rangle}{\sqrt{2}}, \frac{ 00\rangle -  11\rangle}{\sqrt{2}}, \frac{ 01\rangle +  10\rangle}{\sqrt{2}}, \frac{ 01\rangle -  10\rangle}{\sqrt{2}} \right\}$
“Phase” basis	$\left\{ \frac{ 00\rangle +  01\rangle +  10\rangle -  11\rangle}{2}, \frac{ 00\rangle +  01\rangle -  10\rangle +  11\rangle}{2}, \frac{ 00\rangle -  01\rangle +  10\rangle +  11\rangle}{2}, \frac{- 00\rangle +  01\rangle +  10\rangle +  11\rangle}{2} \right\}$

10-III-8

### MOTIVATION 4: FOLLOWING SEQUENCE OF QUANTUM GATE OPERATIONS

$$\begin{array}{c}
 |00\rangle \xrightarrow{R_{y1}(\pi/2)} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{R_{y2}(\pi/2)} \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \\
 \downarrow \text{CNOT} \\
 \frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{R_{y2}(\pi/2)} \frac{|00\rangle + |01\rangle - |10\rangle + |11\rangle}{2}
 \end{array}$$

10-III-9a

## OUTLINE

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10-III-5b

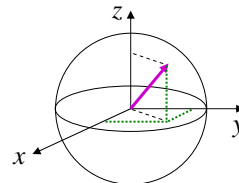
## BLOCH VECTOR REPRESENTATION OF QUBIT DENSITY MATRIX

Qubit state vector:  $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Qubit density matrix  $\rho_{pure} = |\Psi\rangle\langle\Psi|$        $\rho_{mixed} = \sum_r p_r |\Psi_r\rangle\langle\Psi_r|$

Decomposition of density matrix on the basis of the Pauli matrices:

$$\begin{aligned} \rho &= \frac{1}{2}(I + s_x\sigma_x + s_y\sigma_y + s_z\sigma_z) = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma}) \\ &= \frac{1}{2}(I + \langle X \rangle X + \langle Y \rangle Y + \langle Z \rangle Z) \end{aligned}$$



$$\rho \leftrightarrow \vec{s} \quad \langle X \rangle^2 + \langle Y \rangle^2 + \langle Z \rangle^2 \leq 1$$

*we will generalize this notion to a N qubit register*

10-III-10b

## PAULI SPIN MATRICES AND ROTATIONS

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \text{ (identity)}$$

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

$$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

BUILDING BLOCK OF HAMILTONIAN AND MEASUREMENT OPERATORS, THE CLASS OF HERMITIAN OPERATORS. ( $H^\dagger = H$ )

$$[X] = -i\sigma_x = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \rightarrow R_x(\pi)$$

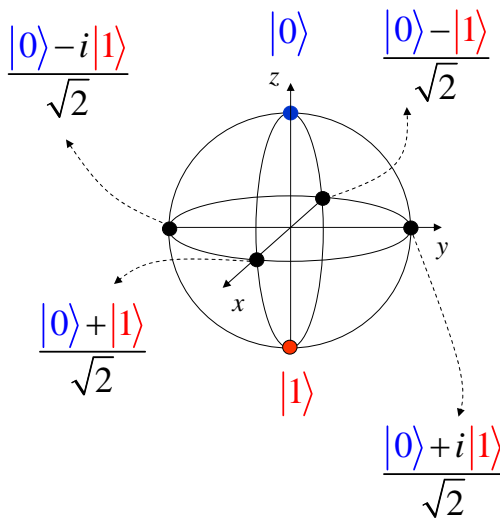
$$[Y] = -i\sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow R_y(\pi)$$

$$[Z] = -i\sigma_z = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \rightarrow R_z(\pi)$$

BUILDING BLOCK OF LOGIC GATE OPERATORS, THE CLASS OF UNITARY OPERATORS. ( $U^\dagger = U^{-1}$ )

10-III-11

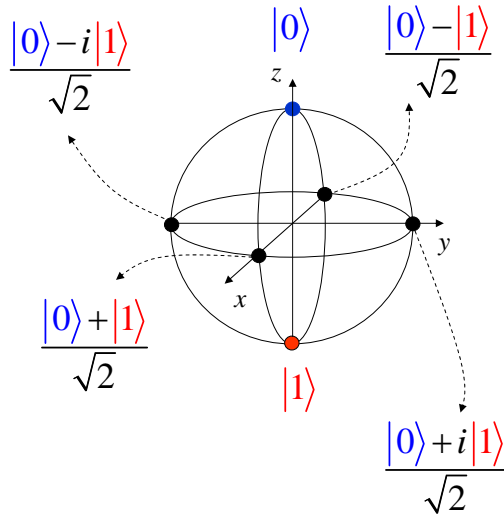
## "CARDINAL POINTS" OF THE BLOCH SPHERE



state	$\langle Z \rangle$	$\langle X \rangle$	$\langle Y \rangle$
$ 0\rangle$	+1	0	0
$ 1\rangle$	-1	0	0
$( 0\rangle +  1\rangle)2^{-1/2}$	0	+1	0
$( 0\rangle -  1\rangle)2^{-1/2}$	0	-1	0
$( 0\rangle + i 1\rangle)2^{-1/2}$	0	0	+1
$( 0\rangle - i 1\rangle)2^{-1/2}$	0	0	-1

10-III-12

## "CARDINAL POINTS" OF THE BLOCH SPHERE

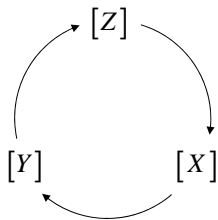


state	$\langle Z \rangle \langle X \rangle \langle Y \rangle$						
$ 0\rangle$	<table border="1"><tr><td>■</td><td></td><td></td></tr><tr><td></td><td></td><td></td></tr></table>	■					
■							
$ 1\rangle$	<table border="1"><tr><td></td><td></td><td></td></tr><tr><td>■</td><td></td><td></td></tr></table>				■		
■							
$2^{-1/2}( 0\rangle+ 1\rangle)$	<table border="1"><tr><td></td><td>■</td><td></td></tr><tr><td></td><td></td><td></td></tr></table>		■				
	■						
$2^{-1/2}( 0\rangle- 1\rangle)$	<table border="1"><tr><td></td><td></td><td></td></tr><tr><td></td><td>■</td><td></td></tr></table>					■	
	■						
$2^{-1/2}( 0\rangle+i 1\rangle)$	<table border="1"><tr><td></td><td></td><td>■</td></tr><tr><td></td><td></td><td></td></tr></table>			■			
		■					
$2^{-1/2}( 0\rangle-i 1\rangle)$	<table border="1"><tr><td></td><td></td><td></td></tr><tr><td></td><td></td><td>■</td></tr></table>						■
		■					

tomogram

10-III-12a

## QUBIT $\pi$ ROTATIONS : QUATERNION GROUP



$$[Z][X] = [Y]$$

and relations  
obtained by circular  
permutations

like:

$$\hat{z} \times \hat{x} = \hat{y}$$

$$[Z]^2 = [X]^2 = [Y]^2 = -I = [Z][X][Y]$$

$$\mathcal{Q} = \{I, -I, [X], [-X], [Y], [-Y], [Z], [-Z]\}$$

8 elements, note "detachable" aspect of minus sign

$$[A]^{-1} = [-A] = -I[A]$$

10-III-13

## GENERALIZED PAULI GROUP DEFINITION

1 qubit:  $\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$

N qubits:  $\mathcal{P}_N = \{+1, -1, +i, -i\} \otimes \{I, Z, X, Y\}^{\otimes N}$

notation:  $I^{\otimes N} = I$

Example of hermitian Pauli group element for 6 qubits:  $-ZIXXIY$

Example of the matrix corresponding to  
1 element of the 2-qubit Pauli group:

$$ZX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Important property:  $ABA^{-1}B^{-1} = \pm I$

10-III-14b

## HERMITIAN PAULI GROUP ELEMENTS ARE LIKE NUMBERS IN BASE 4

Examples:  $+XXX = +\sigma_x^2 \sigma_x^1 \sigma_x^0$

$$-XYZ = -\sigma_x^2 \sigma_y^1 \sigma_z^0$$

$$+XII = +\sigma_x^2$$

$$-IIZ = -\sigma_z^0$$

sign  $\longrightarrow$   $+YIX = +\sigma_y^2 \sigma_x^0$

$\swarrow$  qubit 2       $\uparrow$  qubit 1       $\searrow$  qubit 0

$I \leftrightarrow 00$
$Z \leftrightarrow 01$
$X \leftrightarrow 10$
$Y \leftrightarrow 11$

10-III-15a

## STABILIZER DEFINITION

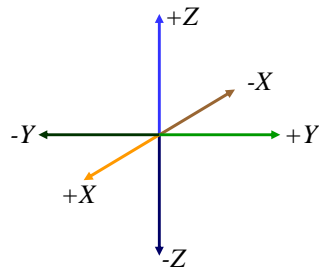
ABELIAN SUBGROUP WHICH DOES NOT CONTAIN  $-I$

1-qubit example:

The 1-qubit Pauli group has 6 stabilizers: we can list them by giving their unique generator, a Pauli operator with a sign.

$$S_{\mathcal{P}}^1 = \{ \{+X\}, \{-X\}, \{+Y\}, \{-Y\}, \{+Z\}, \{-Z\} \}$$

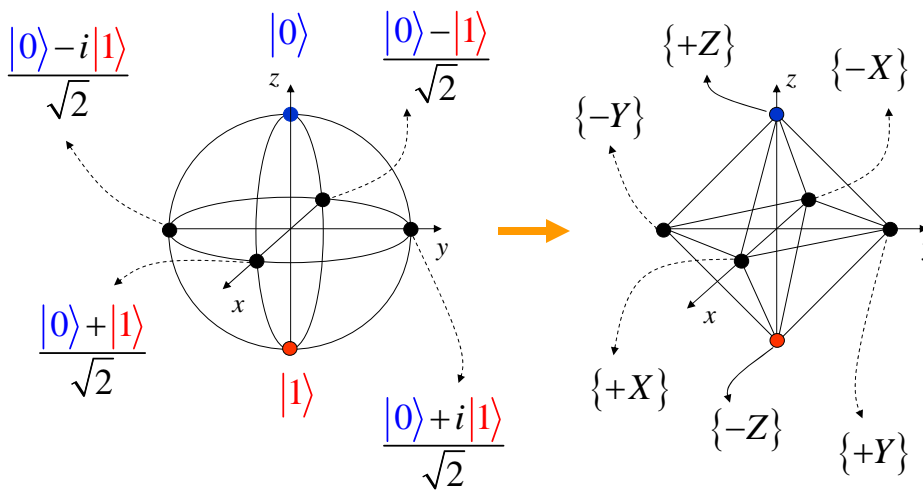
To each generator, we can associate a unit vector in 3d-space:



Each one of these vectors can also be seen as a point of the Bloch sphere, each corresponding to an eigenvector of the element of the stabilizer with eigenvalue 1

10-III-16b

## STABILIZER REPRESENTATION OF "CARDINAL" OR "CLIFFORD" STATES



1 QUBIT IS SIMPLE TO VISUALIZE, WHAT ABOUT N QUBITS?

10-III-17b

## OUTLINE

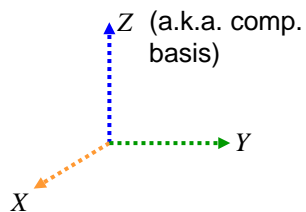
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10-III-5c

## STABILIZER CLASSES FOR 1-QUBIT PAULI GROUP

$$\tilde{S}_P^1 = \{\{Z\}, \{X\}, \{Y\}\} \quad \text{no sign here}$$

To each unsigned Pauli symbol, associate 1 vector of a basis in 3d-space:



Each one of these vectors can be seen as a quantum state basis

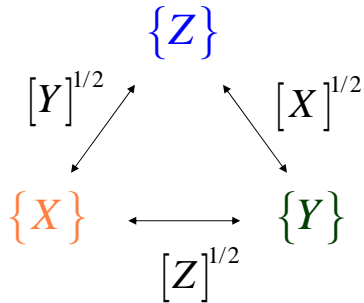
basis states obtained by imposing unity modulus eigenvalues

Stabilizer classes correspond to "independent" bases.

Note differences between "independent" and "orthogonal"

10-III-18b

## $\pi/2$ ROTATIONS LINK STABILIZER CLASSES



The set of isomorphic mappings of the set of stabilizers onto itself is called the Clifford group. For 1-qubit, it is isomorphic to the octahedral group (24 elements).

10-III-19b

## $\pi/2$ ROTATIONS OPERATORS

$$[Z]^{1/2} = \frac{I - iZ}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} \rightarrow R_z(\pi/2)$$

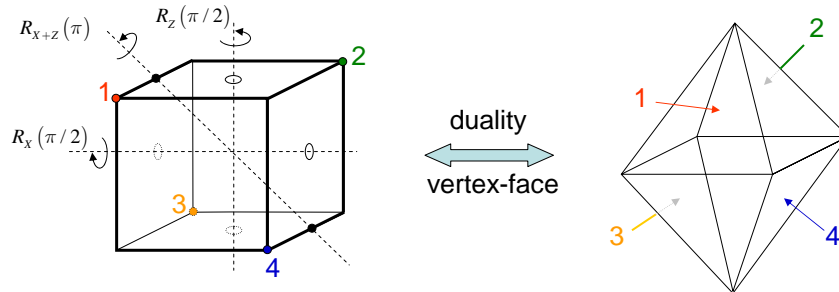
$$[X]^{1/2} = \frac{I - iX}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \rightarrow R_x(\pi/2)$$

$$[Y]^{1/2} = \frac{I - iY}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix} \rightarrow R_y(\pi/2)$$

10-III-20

## GENERATORS OF THE OCTAHEDRAL GROUP (A.K.A. THE 1-QUBIT CLIFFORD GROUP)

Consider  $S_4$  the permutation group on 4 objects, isomorphic to the octahedral group, symmetry group of the cube and the octahedron



1st choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} : [Z]^{1/2}$$

$$R_{X+Z}(\pi) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} : H$$

2nd choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} : [Z]^{1/2}$$

$$R_X(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} : [X]^{1/2}$$

10-III-21c

## OUTLINE

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10-III-5d



<b>KET vs STABILIZER FORMALISMS</b>		
BASIS NAME	KET SET	STABILIZER CLASS
Computational	$\{ 00\rangle,  01\rangle,  10\rangle,  11\rangle\}$	$\{IZ, ZI, ZZ\}$
Bell	$\left\{ \frac{ 00\rangle +  11\rangle}{\sqrt{2}}, \frac{ 00\rangle -  11\rangle}{\sqrt{2}}, \frac{ 01\rangle +  10\rangle}{\sqrt{2}}, \frac{ 01\rangle -  10\rangle}{\sqrt{2}} \right\}$	$\{ZZ, XX, YY\}$
"i-Bell"	$\left\{ \frac{ 00\rangle + i 11\rangle}{\sqrt{2}}, \frac{ 00\rangle - i 11\rangle}{\sqrt{2}}, \frac{ 01\rangle + i 10\rangle}{\sqrt{2}}, \frac{ 01\rangle - i 10\rangle}{\sqrt{2}} \right\}$	$\{ZZ, XY, YX\}$
Phase	$\left\{ \frac{ 00\rangle +  01\rangle +  10\rangle -  11\rangle}{2}, \frac{ 00\rangle +  01\rangle -  10\rangle +  11\rangle}{2}, \frac{ 00\rangle -  01\rangle +  10\rangle +  11\rangle}{2}, \frac{- 00\rangle +  01\rangle +  10\rangle +  11\rangle}{2} \right\}$	$\{ZX, XZ, YY\}$
STABILIZERS ARE MORE COMPACT AND SYMMETRIC, A LANGUAGE MORE ADAPTED TO OPERATIONS		

10-III-24a

### CONDENSED NOTATION FOR CLIFFORD STATES

$\{+IZ, -ZI, -ZZ\} \longrightarrow$

$+IZ$   
 $-ZI$

$(|10\rangle)$

$\{-ZZ, -XX, -YY\} \longrightarrow$

$-ZZ$   
 $-XX$

$\left( \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right)$

Rule: choose N independent generators.  
A Clifford state of a N qubit register is thus N Pauli numbers with N+1 symbols each!

"GHZ" state\*

 $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ 

↓

$\{+IZZ, +ZIZ, +ZZI, +XXX, -XYY, -YXY, -YYX\} \longrightarrow$

$+IZZ$   
 $+ZIZ$   
 $+XXX$

\* Greenberger, Horne and Zeilinger, arXiv:0712.0921

10-III-25a

## STABILIZER AS BIT VALUE TABLE

classical

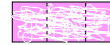
0 1 0

$+ZI$

$-IZ$

$+IZ$

quantum



$+IZZ$

$+ZIZ$

$+XXX$

We have "generalized" classical bit value table, allowing  $X$  and  $Y$  symbols, which do not commute with  $Z$ , in table.

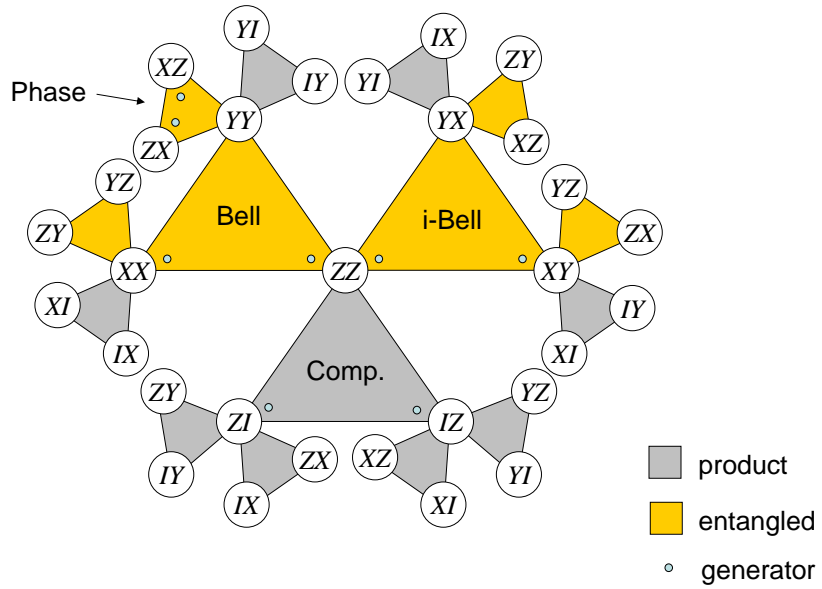
10-III-26b

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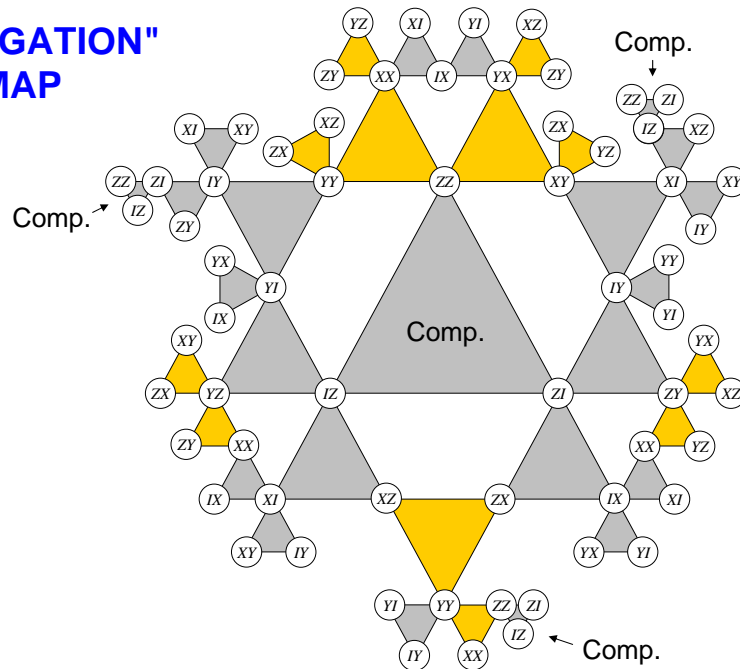
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## MAP OF THE 15 STABILIZER CLASSES



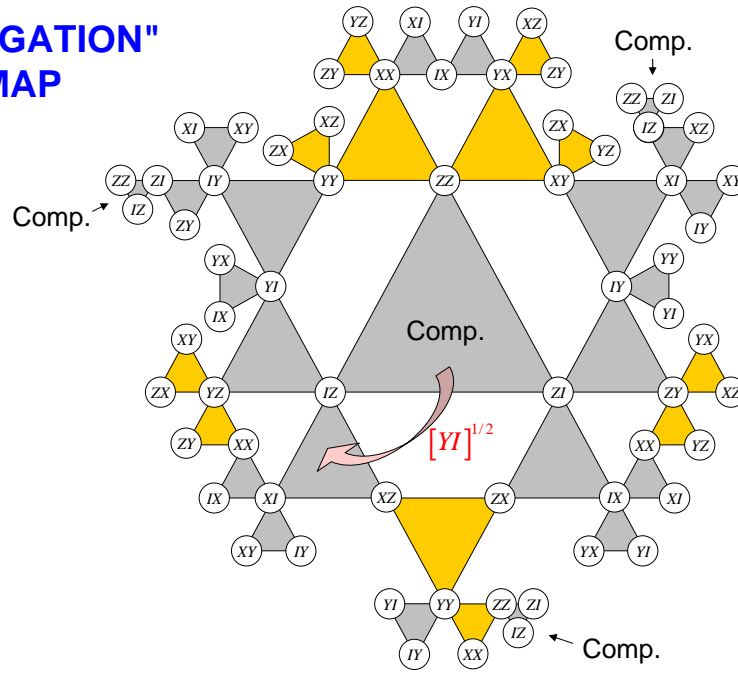
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## "NAVIGATION" MAP



10-III-28

# "NAVIGATION" MAP



10-III-28a

END OF LECTURE