COLLÈGE DE FRANCE 1530

Chaire de Physique Mésoscopique
Michel Devoret
Année 2010, 11 mai - 22 juin

INTRODUCTION AU CALCUL QUANTIQUE

INTRODUCTION TO QUANTUM COMPUTATION

Troisième Leçon / *Third Lecture*

VISIT THE WEBSITE OF THE CHAIR OF MESOSCOPIC PHYSICS

http://www.college-de-france.fr

then follow
Enseignement > Sciences Physiques > Physique Mésoscopique > Site web

or

http://www.physinfo.fr/lectures.html

PDF FILES OF ALL PAST LECTURES ARE POSTED

Questions, comments and corrections are welcome!

write to "phymeso@gmail.com"

CONTENT OF THIS YEAR'S LECTURES

QUANTUM COMPUTATION FROM THE PERSPECTIVE OF MESOSCOPIC CIRCUITS

1. Introduction, c-bits versus q-bits
2. The Pauli matrices and quantum computation primitives
3. <u>Stabilizer formalism for state representation</u>
4. Clifford calculus
5. Algorithms
6. Error correction

10-III-3

CALENDAR OF SEMINARS

May 11: Cristian Urbina, (Quantronics group, SPEC-CEA Saclay)
Josephson effect in atomic contacts and carbon nanotubes

May 18: Benoît Douçot (LPTHE / Université Pierre et Marie Curie)
Towards the physical realization of topologically protected qubits

June 1: Takis Kontos (LPA / Ecole Normale Supérieure)
Points quantiques et ferromagnétisme

June 8: Cristiano Ciuti (MPQ, Université Paris - Diderot)
Ultrastrong coupling circuit QED : vacuum degeneracy and quantum phase transitions

June 15: Leo DiCarlo (Yale)
Preparation and measurement of tri-partite entanglement in a superconducting quantum circuit

June 22: Vladimir Manucharian (Yale)
The fluxonium circuit: an electrical dual of the Cooper-pair box?

10-III-4

LECTURE III : STABILIZER FORMALISM FOR STATE REPRESENTATION

1. Motivations
2. Pauli group and stabilizer definition
3. Stabilizer classes for 1 qubit
4. Stabilizer classes for 2 qubits
5. Stabilizer maps

10-III-5

OUTLINE

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10-III-5a

MOTIVATION 1: DESCRIBING THE CONTENT OF A QUANTUM REGISTER

Example of content of classical 10 bit register:

0 1 0 1 1 0 1 0 0 1

Example of content of quantum 10 bit register (10 qubits):

$$|\Psi\rangle = \frac{1}{2^5}\Big(|0000000000\rangle - |0000000001\rangle + \ldots\ldots$$

$$\ldots\ldots\ldots\ldots + |0101101001\rangle + \ldots\ldots\ldots\ldots$$

$$- |1111111101\rangle + |1111111110\rangle - |1111111111\rangle\Big)$$

in general, 1024 terms!

Complexity of description grows exponentially with the number of qubits....

10-III-6b

MOTIVATION 2: DISTINGUISHISING ENTANGLED FROM PRODUCT STATES

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$: PRODUCT STATE $$\left(= \frac{(|0\rangle + |1\rangle)_1 (|0\rangle + |1\rangle)_2}{2} \right)$$

$$\frac{|00\rangle + |01\rangle - |10\rangle - |11\rangle}{2}$$: PRODUCT STATE $$\left(= \frac{(|0\rangle - |1\rangle)_1 (|0\rangle + |1\rangle)_2}{2} \right)$$

$$\frac{|00\rangle + |01\rangle - |10\rangle + |11\rangle}{2}$$: ENTANGLED STATE

10-III-7a

MOTIVATION 3: IMPROVE SYMMETRY OF BASES

BASIS NAME	KET SET
Computational basis	$\{\|00\rangle, \|01\rangle, \|10\rangle, \|11\rangle\}$
Bell basis	$\left\{\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}, \frac{\|00\rangle-\|11\rangle}{\sqrt{2}}, \frac{\|01\rangle+\|10\rangle}{\sqrt{2}}, \frac{\|01\rangle-\|10\rangle}{\sqrt{2}}\right\}$
"Phase" basis	$\left\{\frac{\|00\rangle+\|01\rangle+\|10\rangle-\|11\rangle}{2}, \frac{\|00\rangle+\|01\rangle-\|10\rangle+\|11\rangle}{2}, \frac{\|00\rangle-\|01\rangle+\|10\rangle+\|11\rangle}{2}, \frac{-\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle}{2}\right\}$

10-III-8

MOTIVATION 4: FOLLOWING SEQUENCE OF QUANTUM GATE OPERATIONS

$$|00\rangle \xrightarrow{R_{y1}(\pi/2)} \frac{|00\rangle+|10\rangle}{\sqrt{2}} \xrightarrow{R_{y2}(\pi/2)} \frac{|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2}$$

$$\frac{|00\rangle+|10\rangle}{\sqrt{2}} \xrightarrow{CNOT} \frac{|00\rangle+|11\rangle}{\sqrt{2}} \xrightarrow{R_{y2}(\pi/2)} \frac{|00\rangle+|01\rangle-|10\rangle+|11\rangle}{2}$$

10-III-9a

OUTLINE

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10-III-5b

BLOCH VECTOR REPRESENTATION OF QUBIT DENSITY MATRIX

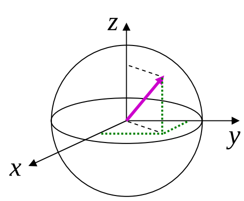
Qubit state vector: $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Qubit density matrix $\rho_{pure} = |\Psi\rangle\langle\Psi|$ $\rho_{mixed} = \sum_r p_r |\Psi_r\rangle\langle\Psi_r|$

Decomposition of density matrix on the basis of the Pauli matrices:

$$\rho = \frac{1}{2}\left(I + s_x\sigma_x + s_y\sigma_y + s_z\sigma_z\right) = \frac{1}{2}\left(I + \vec{s}\cdot\vec{\sigma}\right)$$

$$= \frac{1}{2}\left(I + \langle X\rangle X + \langle Y\rangle Y + \langle Z\rangle Z\right)$$



$$\rho \leftrightarrow \vec{s} \qquad \langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 \leq 1$$

we will generalize this notion to a N qubit register

10-III-10b

PAULI SPIN MATRICES AND ROTATIONS

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \ (\text{identity})$$

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

$$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

BUILDING BLOCK OF HAMILTONIAN AND MEASUREMENT OPERATORS, THE CLASS OF HERMITIAN OPERATORS. $\left(H^\dagger = H\right)$

$$[X] = -i\sigma_x = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \rightarrow R_x(\pi)$$

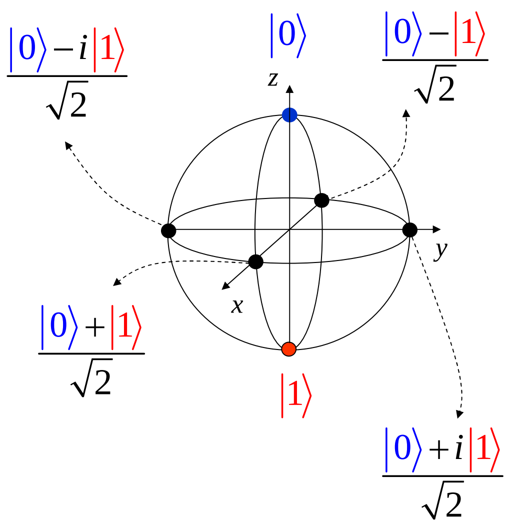
$$[Y] = -i\sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow R_y(\pi)$$

$$[Z] = -i\sigma_z = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \rightarrow R_z(\pi)$$

BUILDING BLOCK OF LOGIC GATE OPERATORS, THE CLASS OF UNITARY OPERATORS. $\left(U^\dagger = U^{-1}\right)$

10-III-11

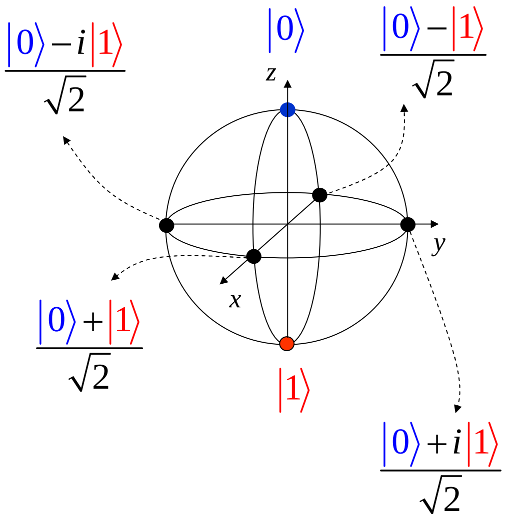
"CARDINAL POINTS" OF THE BLOCH SPHERE



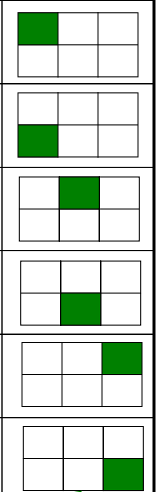
state	$\langle Z\rangle$	$\langle X\rangle$	$\langle Y\rangle$
$\lvert 0\rangle$	+1	0	0
$\lvert 1\rangle$	-1	0	0
$(\lvert 0\rangle + \lvert 1\rangle)2^{-1/2}$	0	+1	0
$(\lvert 0\rangle - \lvert 1\rangle)2^{-1/2}$	0	-1	0
$(\lvert 0\rangle + i\lvert 1\rangle)2^{-1/2}$	0	0	+1
$(\lvert 0\rangle - i\lvert 1\rangle)2^{-1/2}$	0	0	-1

10-III-12

"CARDINAL POINTS" OF THE BLOCH SPHERE

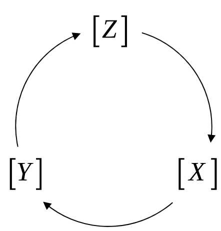


state	$\langle Z\rangle \langle X\rangle \langle Y\rangle$
$|0\rangle$	
$|1\rangle$	
$2^{-1/2}(|0\rangle + |1\rangle)$	
$2^{-1/2}(|0\rangle - |1\rangle)$	
$2^{-1/2}(|0\rangle + i|1\rangle)$	
$2^{-1/2}(|0\rangle - i|1\rangle)$	



tomogram

QUBIT π ROTATIONS : QUATERNION GROUP



$$[Z][X] = [Y]$$

and relations obtained by circular permutations

like:
$\hat{z} \times \hat{x} = \hat{y}$

$$[Z]^2 = [X]^2 = [Y]^2 = -I = [Z][X][Y]$$

$$\mathcal{Q} = \{I, -I, [X], [-X], [Y], [-Y], [Z], [-Z]\}$$

8 elements, note "detachable" aspect of minus sign

$$[A]^{-1} = [-A] = -I[A]$$

GENERALIZED PAULI GROUP DEFINITION

1 qubit: $\mathbb{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$

N qubits: $\mathbb{P}_N = \{+1, -1, +i, -i\} \otimes \{I, Z, X, Y\}^{\otimes N}$

notation: $I^{\otimes N} = I$

Example of hermitian Pauli group element for 6 qubits: $-ZIXXIY$

Example of the matrix corresponding to 1 element of the 2-qubit Pauli group:

$$ZX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Important property: $ABA^{-1}B^{-1} = \pm I$

10-III-14b

HERMITIAN PAULI GROUP ELEMENTS ARE LIKE NUMBERS IN BASE 4

Examples:

$$+XXX = +\sigma_x^2 \sigma_x^1 \sigma_x^0$$

$$-XYZ = -\sigma_x^2 \sigma_y^1 \sigma_z^0$$

$$+XII = +\sigma_x^2$$

$$-IIZ = -\sigma_z^0$$

$$+YIX = +\sigma_y^2 \sigma_x^0$$

sign → ; qubit 2, qubit 1, qubit 0

I	$\leftrightarrow$ 00
Z	$\leftrightarrow$ 01
X	$\leftrightarrow$ 10
Y	$\leftrightarrow$ 11

10-III-15a

STABILIZER DEFINITION

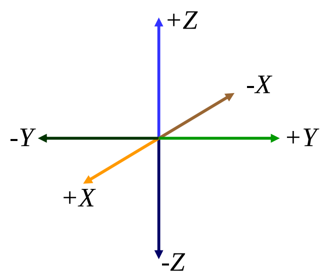
ABELIAN SUBGROUP WHICH DOES NOT CONTAIN ***-I***

1-qubit example:

The 1-qubit Pauli group has 6 stabilizers: we can list them by giving their unique generator, a Pauli operator with a sign.

$$S^1_{\mathbb{P}} = \{\{+X\},\{-X\},\{+Y\},\{-Y\},\{+Z\},\{-Z\}\}$$

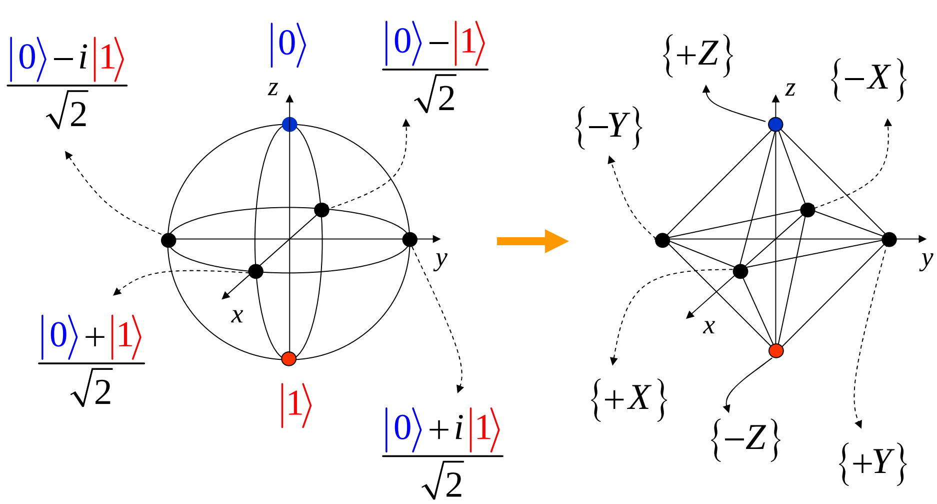
To each generator, we can associate a unit vector in 3d-space:



Each one of these vectors can also be seen as a point of the Bloch sphere, each corresponding to an eigenvector of the element of the stabilizer with eigenvalue 1

10-III-16b

STABILIZER REPRESENTATION OF "CARDINAL" OR "CLIFFORD" STATES



1 QUBIT IS SIMPLE TO VISUALIZE, WHAT ABOUT N QUBITS?

10-III-17b

OUTLINE

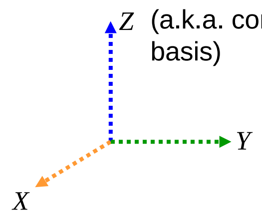
10-III-5c

STABILIZER CLASSES FOR 1-QUBIT PAULI GROUP

$$\tilde{S}^1_{\mathbb{P}} = \{\{Z\},\{X\},\{Y\}\}$$

no sign here

To each unsigned Pauli symbol, associate 1 vector of a basis in 3d-space:



(a.k.a. comp. basis)

Each one of these vectors can be seen as a <u>quantum state</u> basis

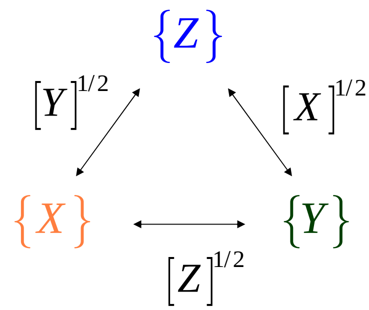
basis states obtained by imposing unity modulus eigenvalues

Stabilizer classes correspond to "independent" bases.

Note differences between "independent" and "orthogonal"

10-III-18b

π/2 ROTATIONS LINK STABILIZER CLASSES



The set of isomorphic mappings of the set of stabilizers onto itself is called the Clifford group. For 1-qubit, it is isomorphic to the octahedral group (24 elements).

10-III-19b

π/2 ROTATIONS OPERATORS

$$[Z]^{1/2} = \frac{I - iZ}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} \rightarrow R_z(\pi/2)$$

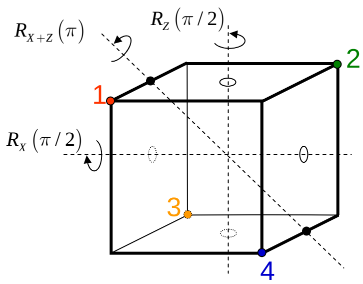
$$[X]^{1/2} = \frac{I - iX}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \rightarrow R_x(\pi/2)$$

$$[Y]^{1/2} = \frac{I - iY}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix} \rightarrow R_y(\pi/2)$$

10-III-20

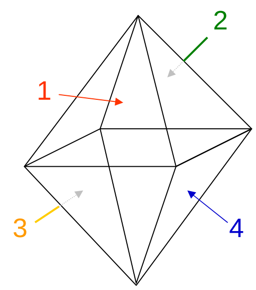
GENERATORS OF THE OCTAHEDRAL GROUP (A.K.A. THE 1-QUBIT CLIFFORD GROUP)

Consider S_4 the permutation group on 4 objects, isomorphic to the octahedral group, symmetry group of the cube and the octahedron



duality

vertex-face



1st choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} : [Z]^{1/2}$$

$$R_{X+Z}(\pi) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} : H$$

2nd choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} : [Z]^{1/2}$$

$$R_X(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} : [X]^{1/2}$$

10-III-21c

OUTLINE

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10-III-5d

THE 2-QUBIT PRIMARY PAULI OPERATORS

right qubit operators

15 +1 primary operators

$$\left\{\begin{matrix} II & IX & IY & IZ \\ XI & XX & XY & XZ \\ YI & YX & YY & YZ \\ ZI & ZX & ZY & ZZ \end{matrix}\right\}$$

left qubit operators

2-qubit operators

10-III-22c

THE 2-QUBIT STABILIZER CLASSES

There are 15* classes of stabilizers with two generators.
We rank them according to the I-Z-X-Y order and obtain:

9 “local” (product states):

$$\left\{\begin{matrix} \{IZ,ZI,ZZ\},\{IZ,XI,XZ\},\{IZ,YI,YZ\},\{IX,ZI,ZX\},\{IX,XI,XX\}, \\ \{IX,YI,YZ\},\{IY,ZI,ZY\},\{IY,XI,XY\},\{IY,YI,YY\} \end{matrix}\right\}$$

6 “global” (entangled states):

$$\left\{\begin{matrix} \{ZZ,XX,YY\},\{ZZ,XY,YX\},\{ZX,XZ,YY\}, \\ \{ZX,XY,YZ\},\{ZY,XX,YZ\},\{ZY,XZ,YX\} \end{matrix}\right\}$$

Each class of stabilizers yields 4 stabilizers since one can change the sign of each one of the two generator. In total, there are 60 stabilizers, each one corresponding to a Clifford state.

*another number 15!

10-III-23a

KET vs STABILIZER FORMALISMS

BASIS NAME	KET SET	STABILIZER CLASS
Computational	$\{\lvert 00\rangle, \lvert 01\rangle, \lvert 10\rangle, \lvert 11\rangle\}$	$\{IZ, ZI, ZZ\}$
Bell	$\left\{\frac{\lvert 00\rangle+\lvert 11\rangle}{\sqrt{2}}, \frac{\lvert 00\rangle-\lvert 11\rangle}{\sqrt{2}}, \frac{\lvert 01\rangle+\lvert 10\rangle}{\sqrt{2}}, \frac{\lvert 01\rangle-\lvert 10\rangle}{\sqrt{2}}\right\}$	$\{ZZ, XX, YY\}$
"i-Bell"	$\left\{\frac{\lvert 00\rangle+i\lvert 11\rangle}{\sqrt{2}}, \frac{\lvert 00\rangle-i\lvert 11\rangle}{\sqrt{2}}, \frac{\lvert 01\rangle+i\lvert 10\rangle}{\sqrt{2}}, \frac{\lvert 01\rangle-i\lvert 10\rangle}{\sqrt{2}}\right\}$	$\{ZZ, XY, YX\}$
Phase	$\left\{\frac{\lvert 00\rangle+\lvert 01\rangle+\lvert 10\rangle-\lvert 11\rangle}{2}, \frac{\lvert 00\rangle+\lvert 01\rangle-\lvert 10\rangle+\lvert 11\rangle}{2}, \frac{\lvert 00\rangle-\lvert 01\rangle+\lvert 10\rangle+\lvert 11\rangle}{2}, \frac{-\lvert 00\rangle+\lvert 01\rangle+\lvert 10\rangle+\lvert 11\rangle}{2}\right\}$	$\{ZX, XZ, YY\}$

STABILIZERS ARE MORE COMPACT AND SYMMETRIC,
A LANGUAGE MORE ADAPTED TO OPERATIONS

10-III-24a

CONDENSED NOTATION FOR CLIFFORD STATES

$$\{+IZ, -ZI, -ZZ\} \longrightarrow \boxed{\begin{matrix}+IZ\\ -ZI\end{matrix}} \qquad (\lvert 10\rangle)$$

$$\{-ZZ, -XX, -YY\} \longrightarrow \boxed{\begin{matrix}-ZZ\\ -XX\end{matrix}} \qquad \left(\frac{\lvert 01\rangle-\lvert 10\rangle}{\sqrt{2}}\right)$$

Rule: choose N independent generators.

A Clifford state of a N qubit register is thus N Pauli numbers with N+1 symbols each!

"GHZ" state* $\frac{1}{\sqrt{2}}(\lvert 000\rangle+\lvert 111\rangle)$

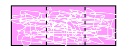
↓

$$\{+IZZ, +ZIZ, +ZZI, +XXX, -XYY, -YXY, -YYX\} \longrightarrow \boxed{\begin{matrix}+IZZ\\ +ZIZ\\ +XXX\end{matrix}}$$

* Greenberger, Horne and Zeilinger, arXiv:0712.0921

10-III-25a

STABILIZER AS BIT VALUE TABLE

classical	quantum
0 1 0	
$+ZII$ $-IZI$ $+IIZ$	$+IZZ$ $+ZIZ$ $+XXX$

We have "generalized" classical bit value table, allowing X and Y symbols, which do not commute with Z, in table.

10-III-26b

OUTLINE

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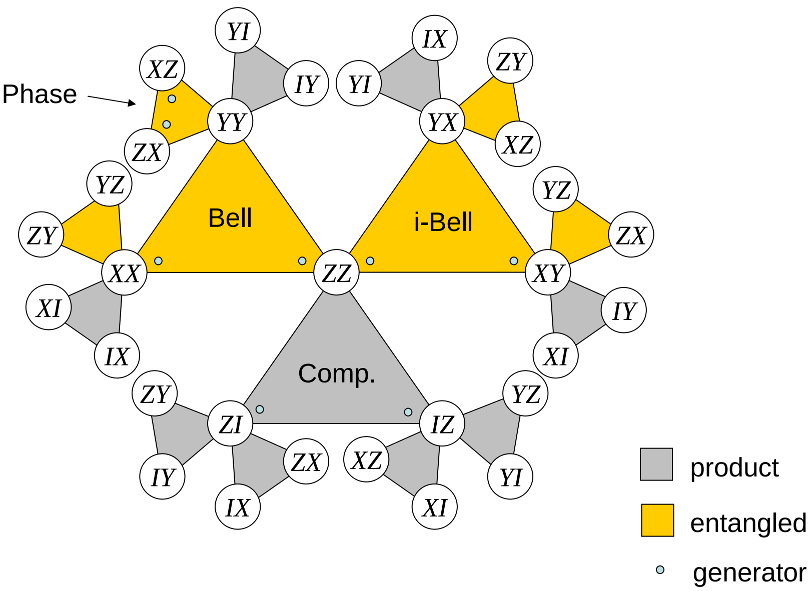
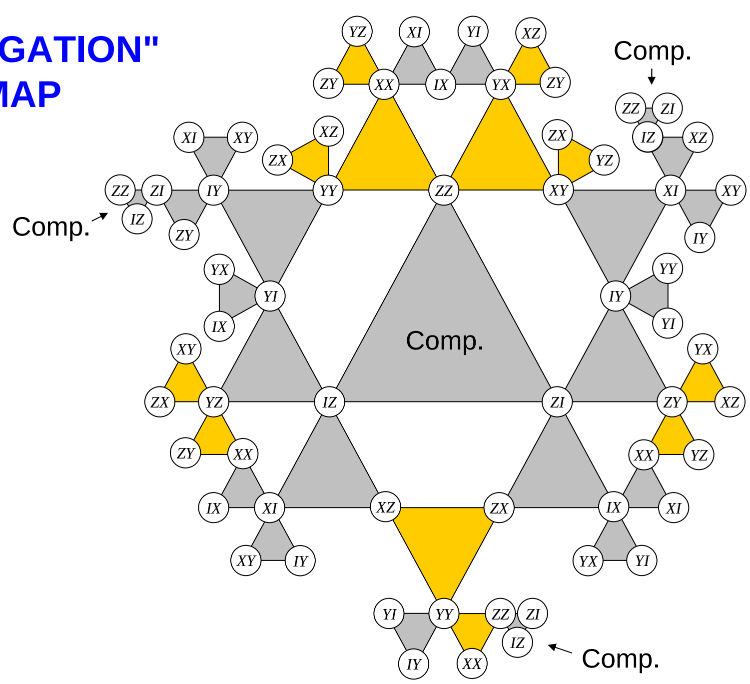
10-III-5e

MAP OF THE 15 STABILIZER CLASSES



10-III-27

"NAVIGATION" MAP



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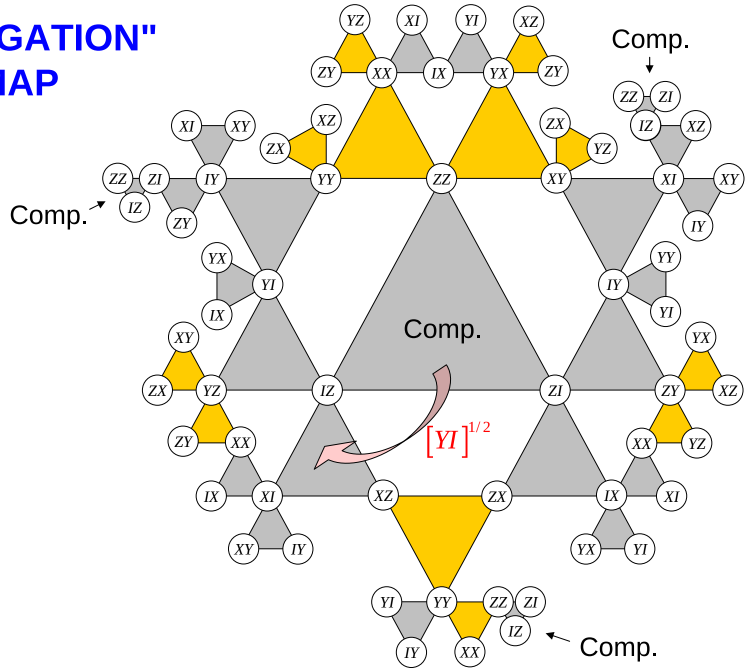
"NAVIGATION" MAP



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END OF LECTURE