



COLLÈGE
DE FRANCE
—1530—



Chaire de Physique Mésoscopique
Michel Devoret
Année 2010, 11 mai - 22 juin

INTRODUCTION AU CALCUL QUANTIQUE
INTRODUCTION TO QUANTUM COMPUTATION

Quatrième Leçon / *Fourth Lecture*

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10-IV-1

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or

<http://www.physinfo.fr/lectures.html>

[PDF FILES OF ALL PAST LECTURES ARE POSTED](#)

Questions, comments and corrections are welcome!

write to "phymeso@gmail.com"

10-IV-2

CONTENT OF THIS YEAR'S LECTURES

QUANTUM COMPUTATION FROM THE PERSPECTIVE OF MESOSCOPIC CIRCUITS

1. Introduction, c-bits versus q-bits
2. The Pauli matrices and quantum computation primitives
3. Stabilizer formalism for state representation
4. Clifford calculus
5. Algorithms
6. Error correction

10-IV-3

CALENDAR OF SEMINARS

May 11: Cristian Urbina, (Quantronics group, SPEC-CEA Saclay)

Josephson effect in atomic contacts and carbon nanotubes

May 18: Benoît Douçot (LPTHE / Université Pierre et Marie Curie)

Towards the physical realization of topologically protected qubits

June 1: Takis Kontos (LPA / Ecole Normale Supérieure)

Points quantiques et ferromagnétisme

June 8: Cristiano Ciuti (MPQ, Université Paris - Diderot)

Ultrastrong coupling circuit QED : vacuum degeneracy and quantum phase transitions

June 15: Leo DiCarlo (Yale)

Preparation and measurement of tri-partite entanglement in a superconducting quantum circuit

June 22: Vladimir Manucharian (Yale)

The fluxonium circuit: an electrical dual of the Cooper-pair box?

10-IV-4

LECTURE IV : CLIFFORD CALCULUS

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations

10-IV-5

OUTLINE

1. Review of stabilizer properties
2. Number of stabilizers
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4. Two-qubit interactions
5. Two-qubit Clifford operations

10-IV-5a

BASIC INGREDIENTS OF STABILIZER FORMALISM

Primary Pauli operators: $P_1 P_2 \dots P_N$ $P_i \in \{I, Z, X, Y\}$
 N : number of qubits

$N=6$ example: **ZIZIXX**

Stabilizer element: $\pm P_1 P_2 \dots P_N = M \neq -I$

Stabilizer: $\{M_1, M_2, \dots, M_{2^N-1}\} \quad \forall \{j, k\}, M_j M_k M_j M_k = I$
 represents a state of the register

Condensed form: $\{M_\alpha, M_\beta, \dots, M_\nu\} \quad \forall \{\alpha, \beta, \gamma\}, M_\alpha M_\beta \neq \pm M_\gamma$
 $\leftarrow N \text{ elements} \rightarrow$

A D -dimensional state manifold of the register is represented by:

$\{M_\alpha, M_\beta, \dots, M_\mu\} \quad D = N - P$
 $\leftarrow P \text{ elements} \rightarrow$

10-IV-6d

STABILIZER ↔ REGISTER STATE

A single-state stabilizer can be written as an array of symbols:

\pm	P_{11}	P_{12}	\dots	P_{1N}	}	N Pauli words with N+1 symbols arranged in "alphabetical" order (I-Z-X-Y)
\pm	P_{21}	P_{22}	\dots	P_{2N}		
\dots	\dots	\dots	\dots	\dots		
\pm	P_{N1}	P_{N2}	\dots	P_{NN}		

The associated state simultaneously diagonalizes the operator-words with + 1 eigenvalue:

$\begin{matrix} +IZ \\ +ZI \end{matrix}$	$ 00\rangle$	$\begin{matrix} +IX \\ +XI \end{matrix}$	$\frac{(0\rangle+ 1\rangle)(0\rangle+ 1\rangle)}{2} = ++\rangle$	$\begin{matrix} -ZZ \\ -XX \end{matrix}$	$\frac{ 01\rangle- 10\rangle}{\sqrt{2}}$
$\begin{matrix} +IZ \\ -ZI \end{matrix}$	$ 10\rangle$	$\begin{matrix} -IX \\ +XI \end{matrix}$	$\frac{(0\rangle+ 1\rangle)(0\rangle- 1\rangle)}{2} = +-\rangle$	$\begin{matrix} -ZZ \\ +XX \end{matrix}$	$\frac{ 01\rangle+ 10\rangle}{\sqrt{2}}$

10-IV-7d

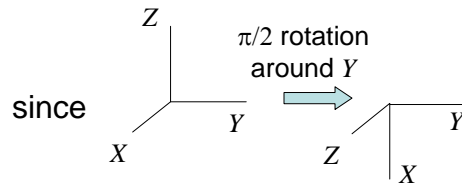
STABILIZER – STATE MAPPING (continued)

$$\begin{array}{|l} +ZZ \\ +XX \end{array} \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)}{2\sqrt{2}} = \frac{|++\rangle - |--\rangle}{\sqrt{2}}$$

remove signs \rightarrow $\begin{array}{|l} ZZ \\ XX \end{array} \rightarrow \left\{ \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right\}$

less is more !

$$\begin{array}{|l} +ZX \\ -XZ \end{array} \rightarrow ? \quad \text{Observe that: } \begin{array}{|l} +ZZ \\ +XX \end{array} \xrightarrow{R_{y_2}\left(\frac{\pi}{2}\right)} \begin{array}{|l} +ZX \\ -XZ \end{array}$$



$$\frac{|00\rangle + |01\rangle - |10\rangle + |11\rangle}{2}$$

10-IV-8e

STABILIZER – STATE MAPPING (continued)

$\begin{array}{ l} +IZ \\ +ZI \end{array}$	$ 00\rangle$	$\begin{array}{ l} +ZZ \\ +XX \end{array}$	$\frac{ 00\rangle + 11\rangle}{\sqrt{2}}$	$\begin{array}{ l} +ZZ \\ -XX \end{array}$	$\frac{ 00\rangle - 11\rangle}{\sqrt{2}}$
$+ZZ$	\leftarrow pivot		$XX 00\rangle = 11\rangle$		
			$XX 11\rangle = 00\rangle$		

$\begin{array}{ l} +IIZ \\ +IZI \\ +ZII \end{array}$	$ 000\rangle$	$\begin{array}{ l} +IZZ \\ +ZIZ \\ +XXX \end{array}$	$\frac{ 000\rangle + 111\rangle}{\sqrt{2}}$	$\begin{array}{ l} +IZZ \\ +ZIZ \\ -XXX \end{array}$	$\frac{ 000\rangle - 111\rangle}{\sqrt{2}}$
$+IZZ$	\leftarrow pivot				
$+ZIZ$	\leftarrow pivot				
$+ZII$			$XXX 000\rangle = 111\rangle$		
$+ZZI$			$XXX 111\rangle = 000\rangle$		

10-IV-9

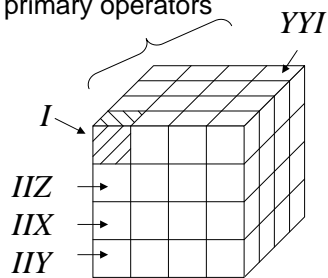
OUTLINE

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations

10-IV-5b

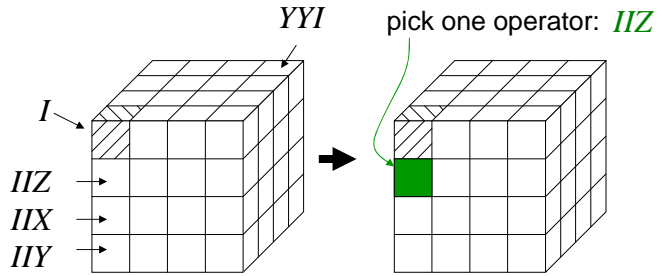
CONSTRUCTION OF A STABILIZER CLASS

All the Pauli multi-qubit
primary operators



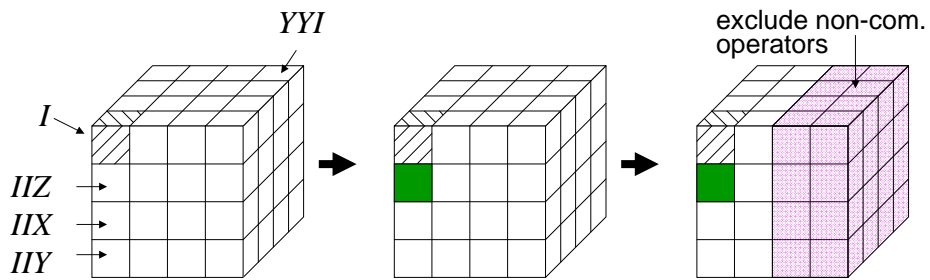
10-IV-10

CONSTRUCTION OF A STABILIZER CLASS



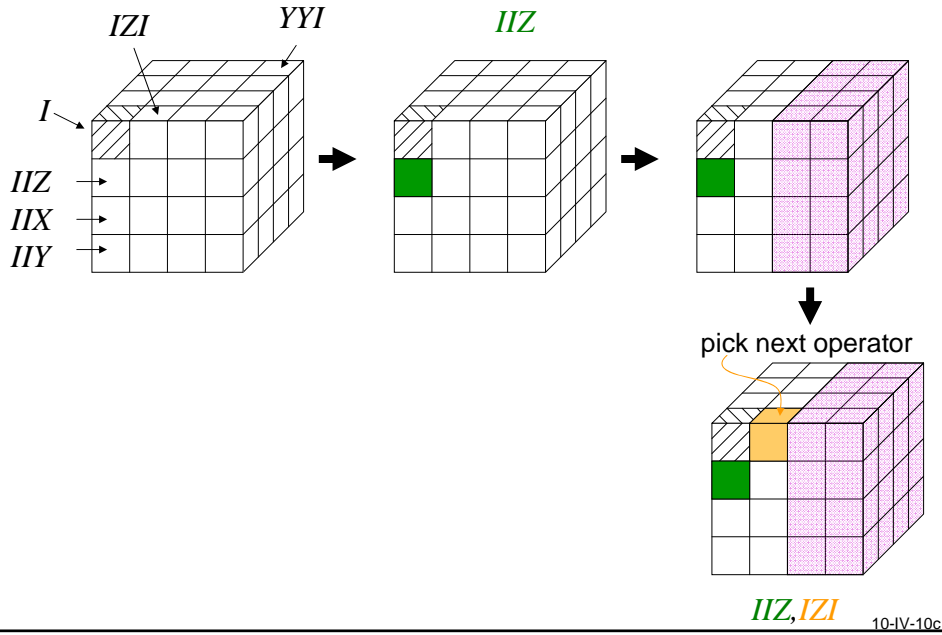
10-IV-10a

CONSTRUCTION OF A STABILIZER CLASS

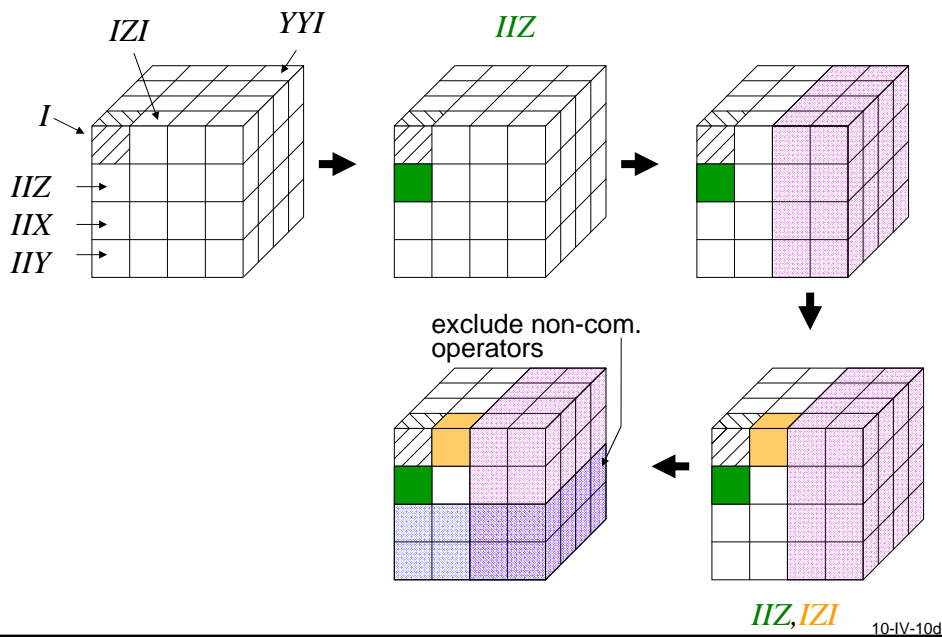


10-IV-10b

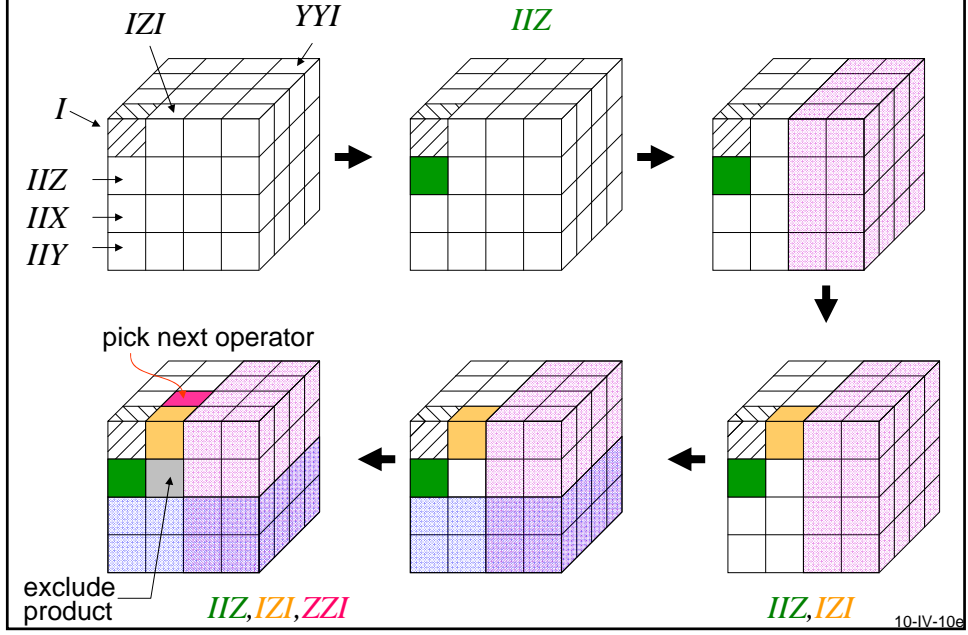
CONSTRUCTION OF A STABILIZER CLASS



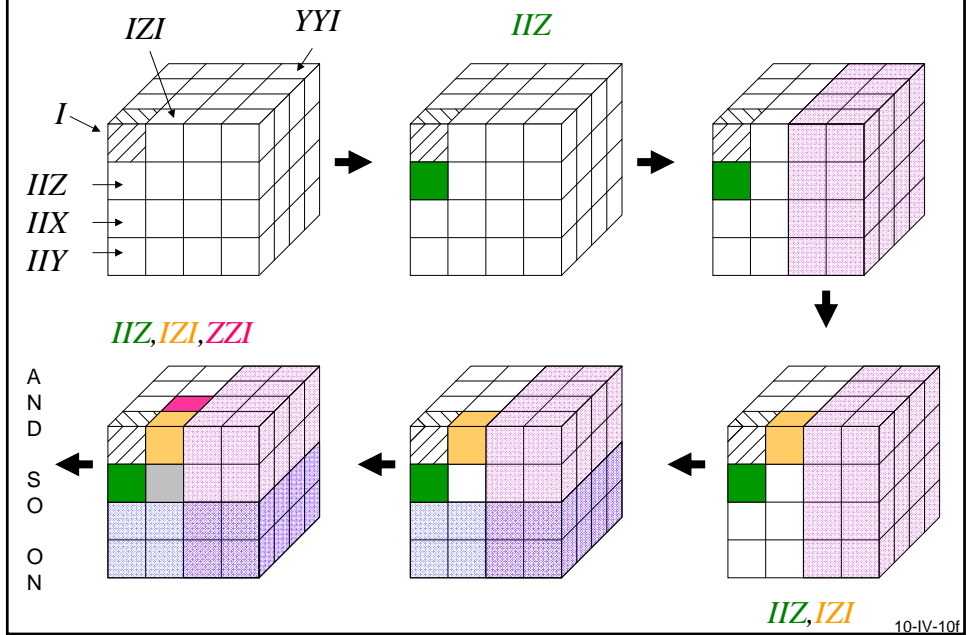
CONSTRUCTION OF A STABILIZER CLASS



CONSTRUCTION OF A STABILIZER CLASS



CONSTRUCTION OF A STABILIZER CLASS



NUMBER OF STABILIZER CLASSES

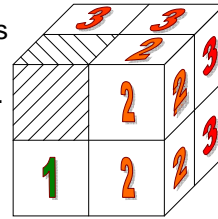
- First choice: $2^{2N} - 1$ the -1 comes from excluding I
- Second choice: $2^{2N-1} - 2$ exclude non.com., last gen. and I
- Third choice: $2^{2N-2} - 4$ exclude non.com., last gen. comb. and I
- ⋮
- k -th choice: $2^{2N-k+1} - 2^{k-1}$

Total nb. choices:
$$\prod_{k=1}^N (2^{2N-k+1} - 2^{k-1}) = \prod_{k=1}^N (2^{N-k+1} + 1)(2^N - 2^{k-1})$$

Last factor corresponds to the number of arrangements of numbers 1 thru N on hyper-cube of stabilizer gen. combinations. It must be divided out to get the classes.

The number of stabilizer classes is thus:

$$\prod_{k=1}^N (2^{N-k+1} + 1) = (2^N + 1)(2^{N-1} + 1) \dots \cdot 5 \cdot 3$$

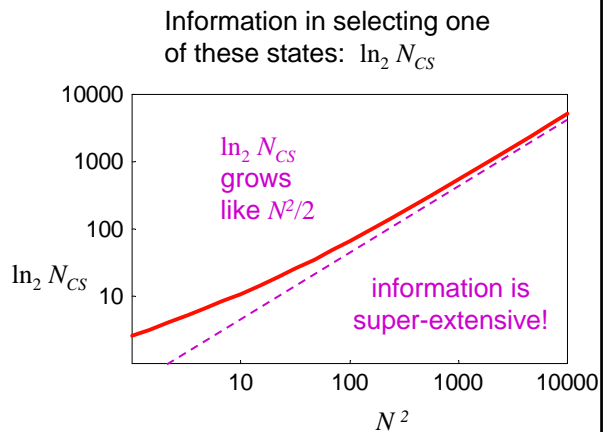


10-IV-11a

NUMBER OF CLIFFORD STATES

$$N_{CS} = 2^N (2^N + 1)(2^{N-1} + 1) \dots \cdot 5 \cdot 3$$

N	N_{CS}
1	6
2	60
3	1080
4	36720
5	2423520



Gottesman-Knill theorem

entanglement, Bell violations, teleportation, error correction, but...

10-IV-12c

OUTLINE

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10-IV-5c

THE SINGLE QUBIT CLIFFORD GROUP

Consider isomorphisms of the Pauli group into itself

Each isomorphism is characterized by the set of the images of the generators: $\{g_z, g_x\} \in \{Z, -Z, X, -X, Y, -Y\}^{\otimes 2}$

$$\begin{cases} Z \mapsto g_z \\ X \mapsto g_x \end{cases} \quad \begin{array}{l} \text{The two images must satisfy:} \\ g_z g_x = -g_x g_z \end{array}$$

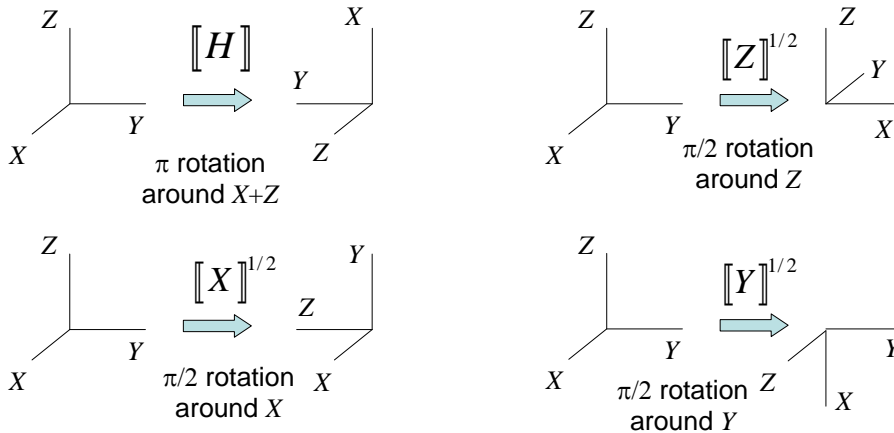
There are therefore $6 \times 4 = 24$ isomorphisms, called elements of the 1-qubit Clifford group:

$$\begin{aligned} Id &= \begin{Bmatrix} Z \mapsto Z \\ X \mapsto X \end{Bmatrix} & [[Z]]^{1/2} &= \begin{Bmatrix} Z \mapsto Z \\ X \mapsto Y \end{Bmatrix} & [[X]]^{1/2} &= \begin{Bmatrix} Z \mapsto -Y \\ X \mapsto X \end{Bmatrix} & [[Y]]^{1/2} &= \begin{Bmatrix} Z \mapsto X \\ X \mapsto -Z \end{Bmatrix} \\ [[H]] &= \begin{Bmatrix} Z \mapsto X \\ X \mapsto Z \end{Bmatrix} & [[Z]]^{-1/2} &= \begin{Bmatrix} Z \mapsto Z \\ X \mapsto -Y \end{Bmatrix} & [[X]]^{-1/2} &= \begin{Bmatrix} Z \mapsto Y \\ X \mapsto X \end{Bmatrix} & [[Y]]^{-1/2} &= \begin{Bmatrix} Z \mapsto -X \\ X \mapsto Z \end{Bmatrix} \\ & & [[Z]] &= \begin{Bmatrix} Z \mapsto Z \\ X \mapsto -X \end{Bmatrix} & [[X]] &= \begin{Bmatrix} Z \mapsto -Z \\ X \mapsto X \end{Bmatrix} & [[Y]] &= \begin{Bmatrix} Z \mapsto -Z \\ X \mapsto -X \end{Bmatrix} \end{aligned}$$

Why $[[P]]^{1/2}$ and not $[P]^{1/2}$? The latter is an operator acting on kets. The former is a super-operator acting on operators.

10-IV-13c

THE SINGLE QUBIT CLIFFORD GROUP IS ISOMORPHIC TO THE OCTAHEDRAL GROUP

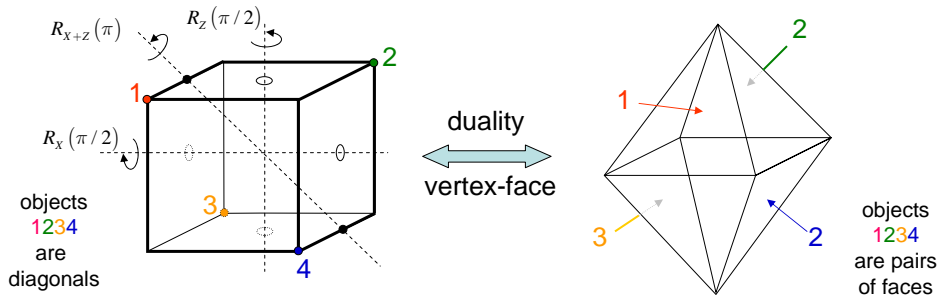


The notation $[[A]]^a$ means the transformation is a "rotation" around axis A with angle $a\pi$. Attention: $[[A]]^2 = Id$

10-IV-14b

GENERATORS OF THE 1-QUBIT CLIFFORD GROUP (ISOMORPHIC TO THE OCTAHEDRAL GROUP)

Consider S_4 the permutation group on 4 objects, isomorphic to the octahedral group, symmetry group of the cube and the octahedron



1st choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = [[Z]]^{1/2}$$

$$R_{X+Z}(\pi) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix} = [[H]]$$

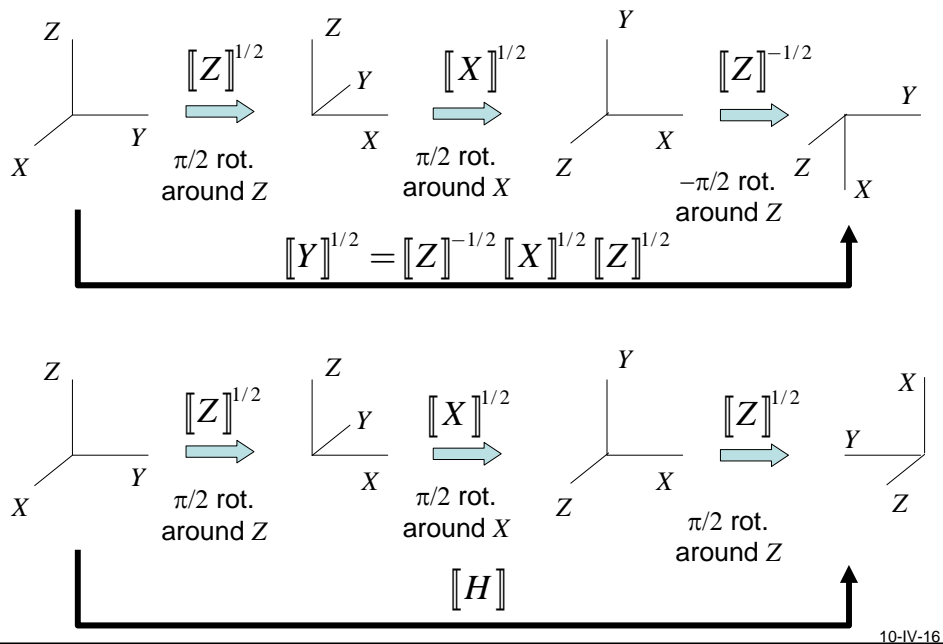
2nd choice of generators

$$R_Z(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = [[Z]]^{1/2}$$

$$R_X(\pi/2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = [[X]]^{1/2}$$

10-IV-15a

USEFUL RELATIONS FOR QUANTUM COMPILERS



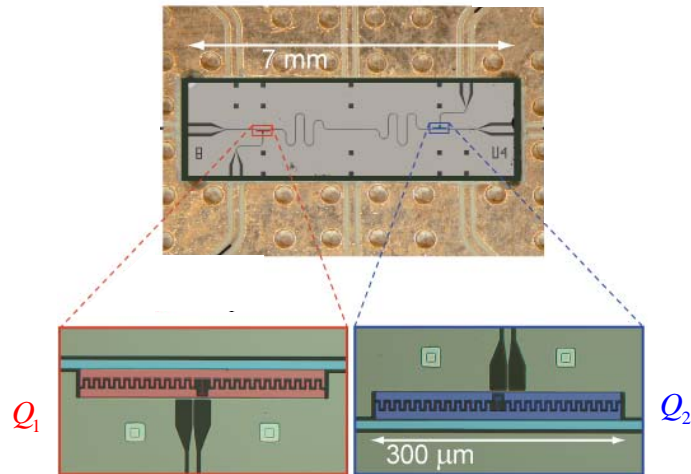
OUTLINE

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10-IV-5d

TWO-QUBIT QUANTUM PROCESSOR

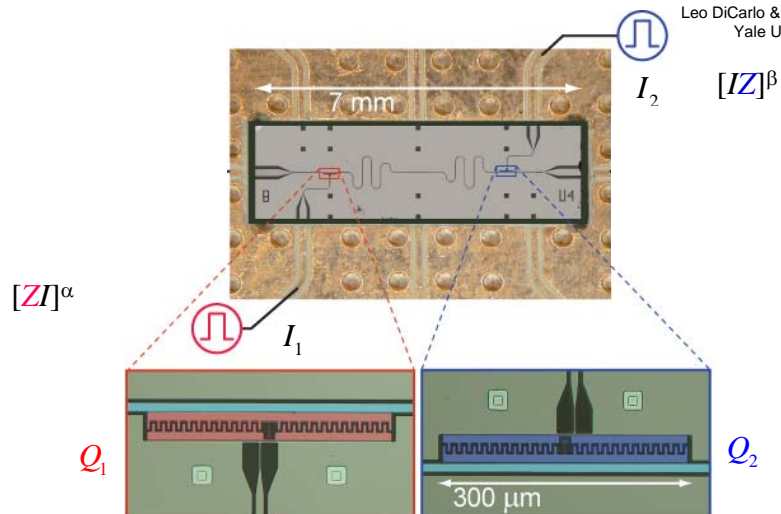
slide courtesy of
Leo DiCarlo & Rob Schoelkopf
Yale University



10-IV-17

TWO-QUBIT QUANTUM PROCESSOR

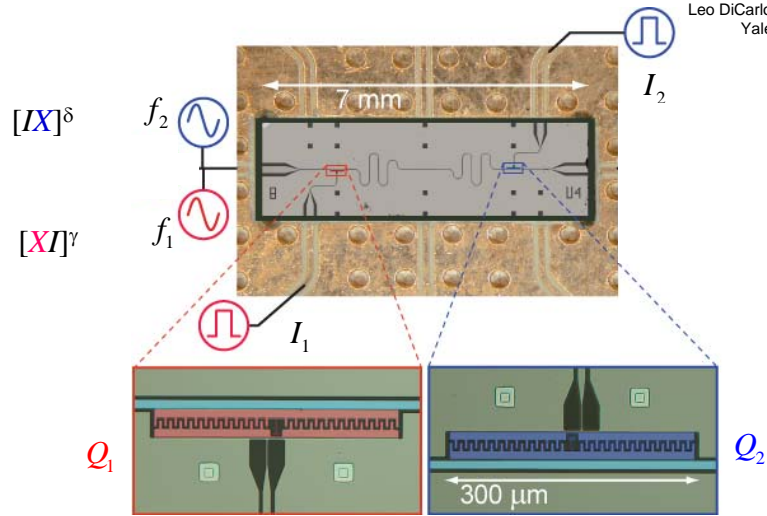
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10-IV-17a

TWO-QUBIT QUANTUM PROCESSOR

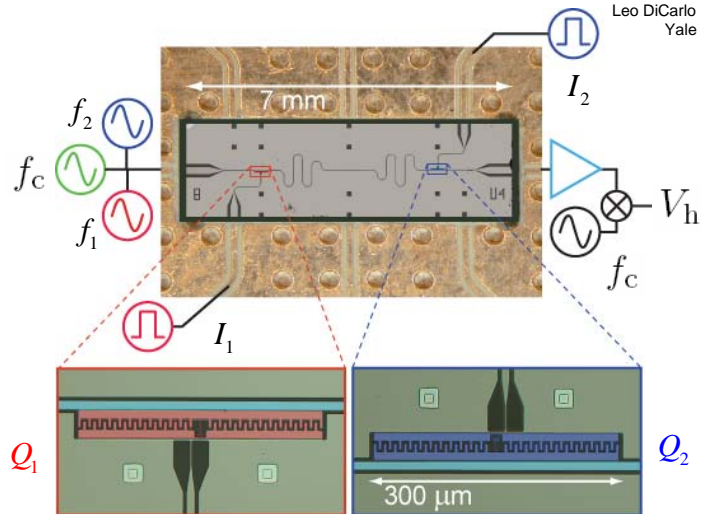
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10-IV-17b

TWO-QUBIT QUANTUM PROCESSOR

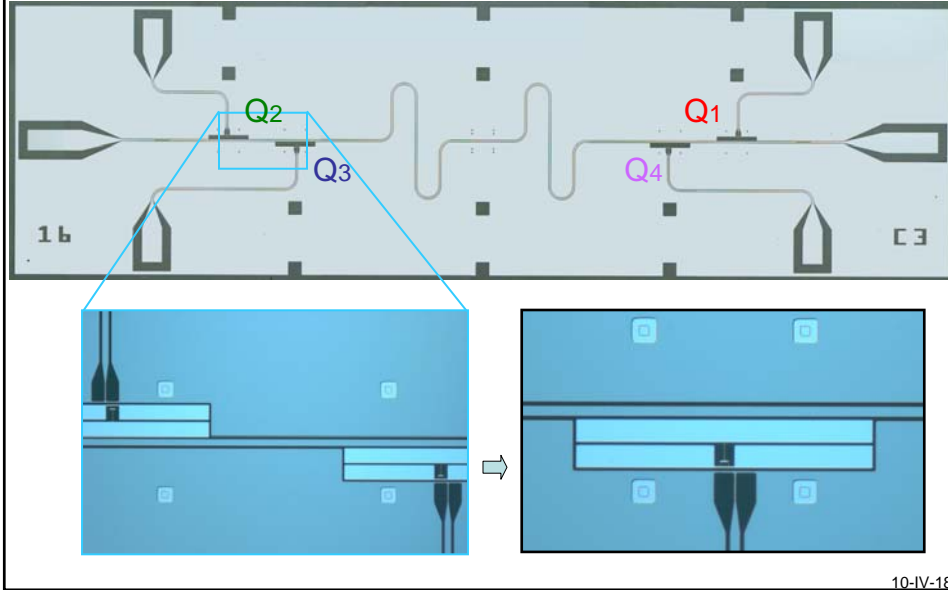
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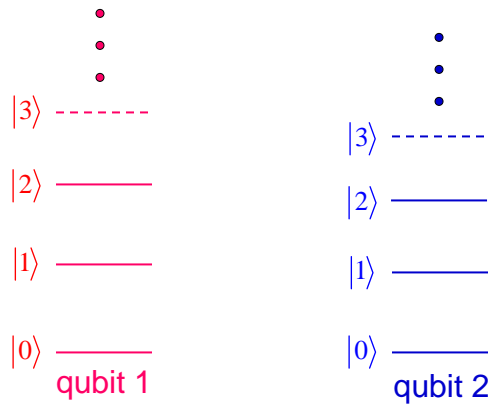
10-IV-17c

UPGRADING THE PROCESSOR TO 4 QUBITS

slide courtesy of Leo DiCarlo & Rob Schoelkopf



CIRCUIT HAMILTONIAN

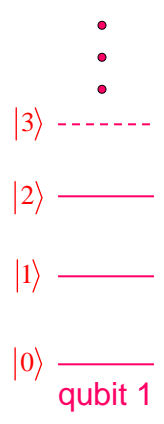


$$\frac{H_1}{\hbar} = \omega_1 a_1^\dagger a_1 + \frac{\alpha_1}{2} (a_1^\dagger a_1)^2$$

$$\frac{H_2}{\hbar} = \omega_2 a_2^\dagger a_2 + \frac{\alpha_2}{2} (a_2^\dagger a_2)^2$$

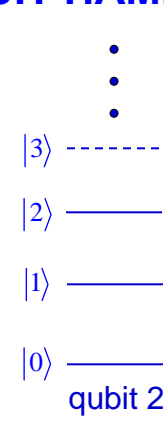
10-IV-19

CIRCUIT HAMILTONIAN



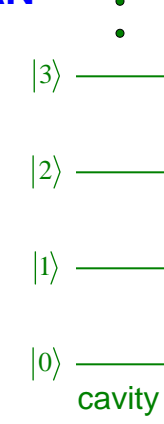
qubit 1

$$\frac{H_1}{\hbar} = \omega_1 a_1^\dagger a_1 + \frac{\alpha_1}{2} (a_1^\dagger a_1)^2$$



qubit 2

$$\frac{H_2}{\hbar} = \omega_2 a_2^\dagger a_2 + \frac{\alpha_2}{2} (a_2^\dagger a_2)^2$$



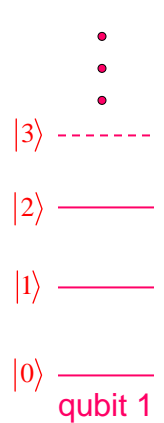
cavity

$$\frac{H_c}{\hbar} = \omega_c a_c^\dagger a_c$$

$H_{\text{total}} = H_1 + H_2 + H_c + H_{\text{coupling}}$

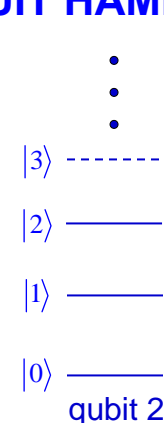
10-IV-19a

CIRCUIT HAMILTONIAN



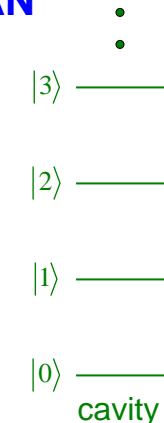
qubit 1

$$\frac{H_1}{\hbar} = \omega_1 a_1^\dagger a_1 + \frac{\alpha_1}{2} (a_1^\dagger a_1)^2$$



qubit 2

$$\frac{H_2}{\hbar} = \omega_2 a_2^\dagger a_2 + \frac{\alpha_2}{2} (a_2^\dagger a_2)^2$$



cavity

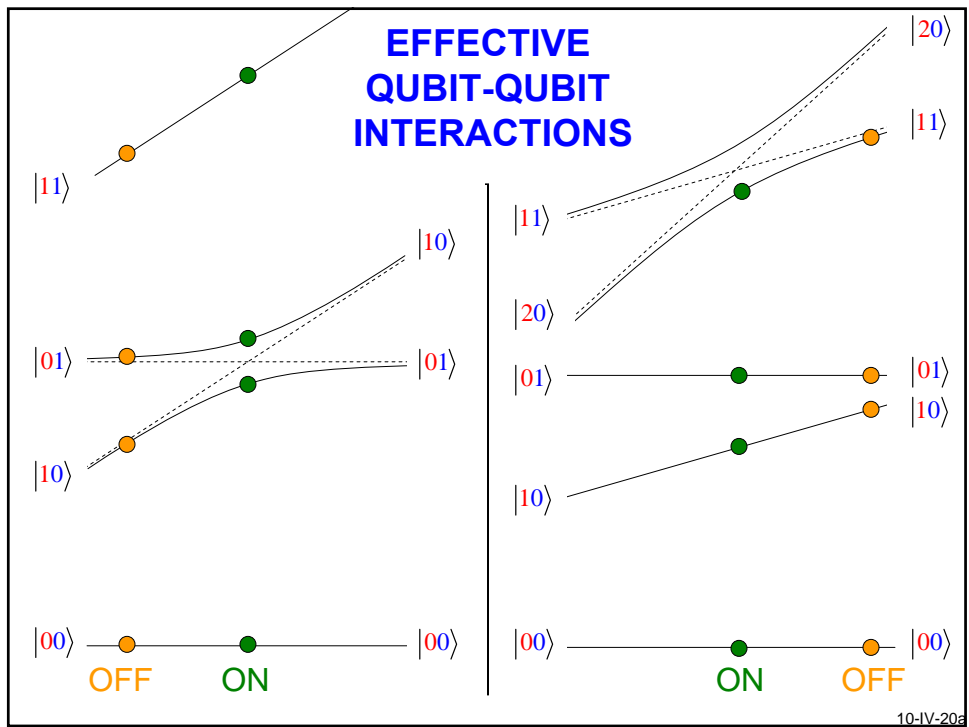
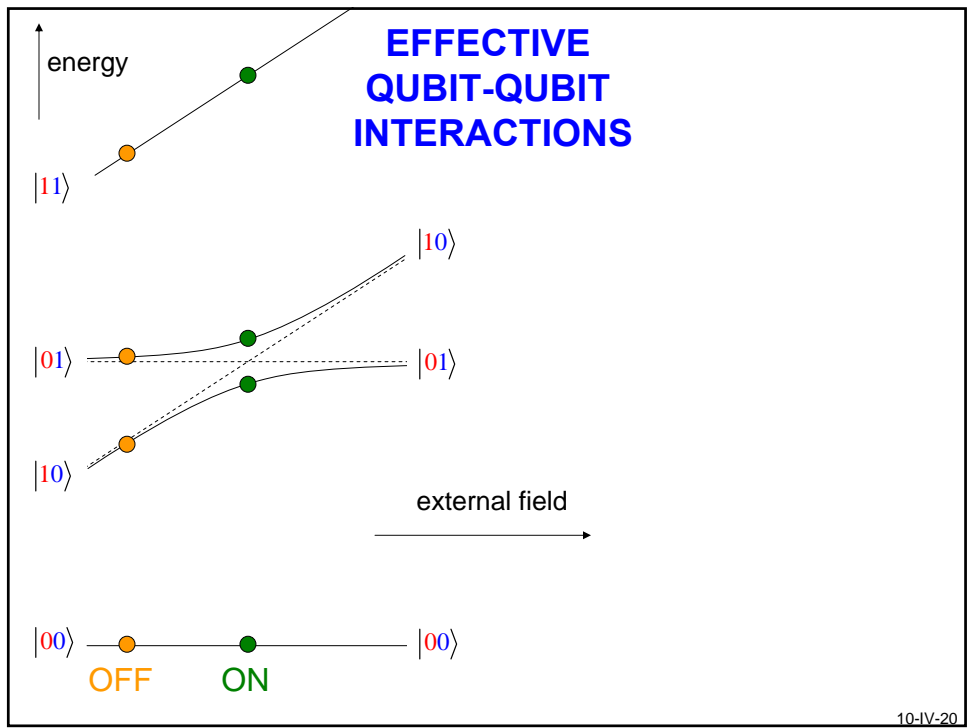
$$\frac{H_c}{\hbar} = \omega_c a_c^\dagger a_c$$

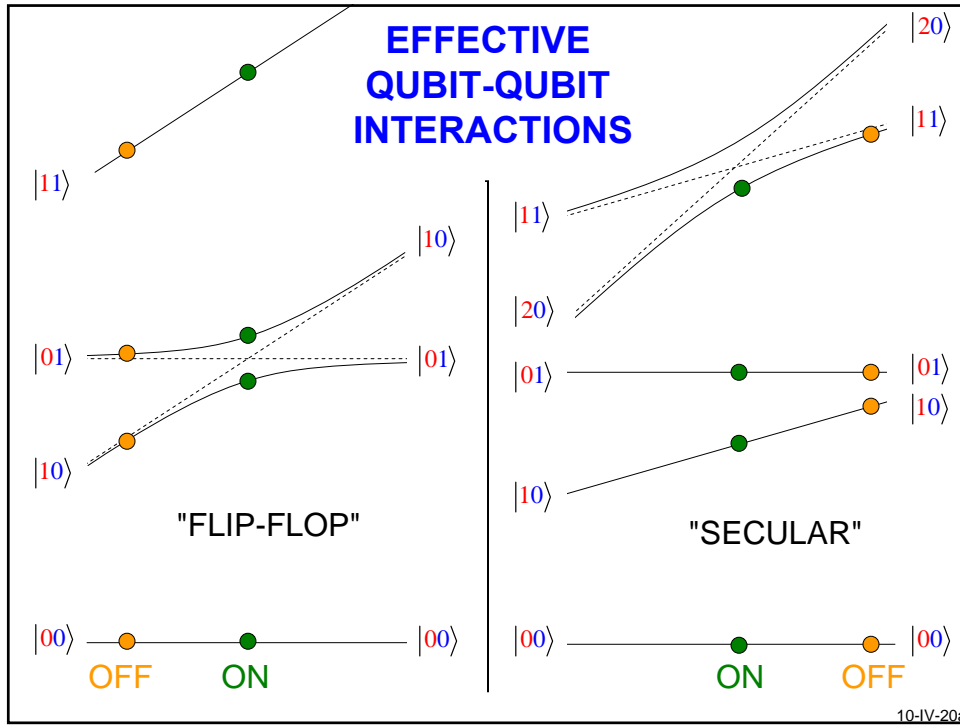
$$\frac{H_{\text{coupling}}}{\hbar} \simeq g_1 (a_1^\dagger a_c + a_1 a_c^\dagger) + g_2 (a_2^\dagger a_c + a_2 a_c^\dagger)$$

RWA

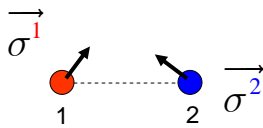
$H_{\text{total}} = H_1 + H_2 + H_c + H_{\text{coupling}}$

10-IV-19b





NATURAL ENTANGLING OPERATIONS



$$\hat{U}(\tau) = \exp(-i\hat{H}_{\text{int}}\tau/\hbar)$$

* Secular interaction: $\hat{H}_{\text{int}} = g_{\parallel}\sigma_z^1\sigma_z^2 \longrightarrow [ZZ]^{1/2}$

with adjustment of gate duration time: $\tau_s = \frac{\pi\hbar}{4g_{\parallel}}$

* Flip-flop interaction: $\hat{H}_{\text{int}} = g_{\perp}(\sigma_+^1\sigma_-^2 + \sigma_-^1\sigma_+^2)$

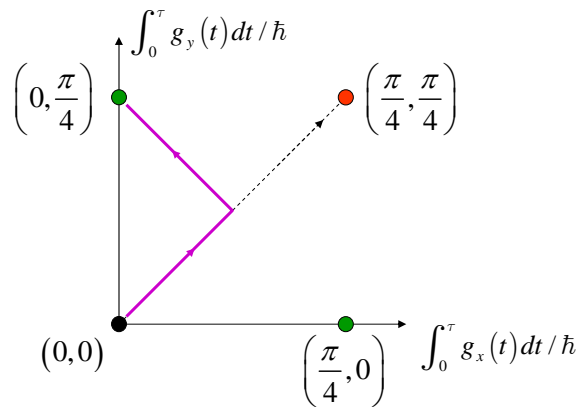
$= g_{\perp}(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2)/2$

$\tau_f = \frac{\pi\hbar}{4g_{\perp}} \longrightarrow [XX]^{1/4} [YY]^{1/4}$

10-IV-21

REFOCUS SEQUENCE NECESSARY WITH FLIP-FLOP INTERACTION

Consider $\hat{H}_{\text{int}} = g_x(t)\sigma_{1x}\sigma_{2x} + g_y(t)\sigma_{1y}\sigma_{2y}$



Green points:
[ZZ]^{1/2} equivalent

Need to cut sequence
in 2 sections, with
a 180° flip of one qubit
around x or y.

10-IV-22

OUTLINE

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10-IV-5e

GOTTESMAN TABLES FOR QUANTUM OPERATIONS

Bit Flip
(NOT)

[X]	
Z	-Z
X	X

π rotation
along \hat{x}

Hadamard

π rotation
along $\hat{x} + \hat{z}$

H	
Z	X
X	Z

Phase Flip

π rotation
along \hat{z}

[Z]	
Z	Z
X	-X

NOT^{1/2}

$\pi/2$ rotation
along \hat{y}

[Y] ^{1/2}	
Z	X
X	-Z

Example of a 2-qubit gate:

$$|c, t\rangle \rightarrow |c, t \oplus c\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

CNOT	
IZ	
ZI	
IX	
XI	

← phase kick-back

} as in Boolean algebra

control ↑ target

10-IV-23a

RULES OF CLIFFORD CALCULUS FOR N QUBITS

They are found starting from: $[[B]]^\alpha A = [B]^{-\alpha} A [B]^\alpha$

$[[B]]^1 [A] = [-A]$	if A and B anticommute
$= [A]$	if A and B commute

$[[B]]^{1/2} [A] = [B][A]$	if A and B anticommute
$= [A]$	if A and B commute

NOTE THAT: $[-B]^\alpha = [[B]]^{-\alpha}$
 $[A] = [B] \Leftrightarrow A = B$

10-IV-24a

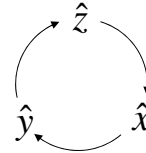
EXAMPLE OF CLIFFORD CALCULATIONS

$$\begin{array}{lll}
 \llbracket IZ \rrbracket^{1/2} IX = IY & \llbracket ZI \rrbracket^{1/2} IX = IX & \llbracket ZI \rrbracket^{1/2} XI = YI \\
 \llbracket IZ \rrbracket^{1/2} IY = -IX & \llbracket YI \rrbracket^{1/2} IZ = IZ & \llbracket ZI \rrbracket^{1/2} YI = -XI \\
 \llbracket IY \rrbracket^{1/2} IZ = IX & \llbracket IZ \rrbracket^{1/2} XI = XI & \llbracket YI \rrbracket^{1/2} ZI = XI \\
 \llbracket IY \rrbracket^{1/2} IX = -IZ & \llbracket IY \rrbracket^{1/2} ZI = ZI & \llbracket YI \rrbracket^{1/2} XI = -ZI
 \end{array}$$

$$\begin{array}{l}
 \llbracket ZZ \rrbracket^{1/2} IX = ZY \\
 \llbracket ZZ \rrbracket^{1/2} XI = YZ \\
 \llbracket ZZ \rrbracket^{1/2} XX = XX \\
 \llbracket ZZ \rrbracket^{1/2} YY = YY
 \end{array}$$

Only need to know:

$$\hat{z} \times \hat{x} = \hat{y}$$



10-IV-25

EXAMPLE OF POWER OF STABILIZER FORMALISM (1)

How do we go from the Computational basis to the Sign basis?

$$\begin{array}{ccc}
 \{IZ, ZI, ZZ\} & \longrightarrow & \{IX, XI, XX\} \\
 \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} & & \{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}
 \end{array}$$

For both qubit, Z must be changed into X
This is performed by a 90° rotation around Y (easier than H).

$$\begin{array}{l}
 \{IZ, ZI, ZZ\} \xrightarrow{\llbracket YI \rrbracket^{1/2} \llbracket IY \rrbracket^{1/2}} \{IX, XI, XX\} \\
 \llbracket IY \rrbracket^{1/2} IZ = IX \\
 \llbracket IY \rrbracket^{1/2} ZI = ZI \\
 \llbracket YI \rrbracket^{1/2} ZI = XI \\
 \llbracket YI \rrbracket^{1/2} IX = IX
 \end{array}$$

10-IV-26a

EXAMPLE OF POWER OF STABILIZER FORMALISM (2)

How do we go from the i-Sign basis to the Phase basis?

$$\{IY, YI, YY\} \longrightarrow \{ZX, XZ, YY\}$$

$$\left\{ \frac{(|0\rangle+i|1\rangle)(|0\rangle+i|1\rangle)}{2}, \frac{(|0\rangle+i|1\rangle)(|0\rangle-i|1\rangle)}{2}, \frac{(|0\rangle-i|1\rangle)(|0\rangle+i|1\rangle)}{2}, \frac{(|0\rangle-i|1\rangle)(|0\rangle-i|1\rangle)}{2} \right\}$$

$$\left\{ \frac{(|00\rangle+|01\rangle+|10\rangle-|11\rangle)}{2}, \frac{(|00\rangle+|01\rangle-|10\rangle+|11\rangle)}{2}, \frac{(|00\rangle-|01\rangle+|10\rangle+|11\rangle)}{2}, \frac{-|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2} \right\}$$

We need an entanglement operator: $[[ZZ]]^{1/2}$

$$[[ZZ]]^{1/2} IY = -ZX$$

$$[[ZZ]]^{1/2} YI = -XZ$$

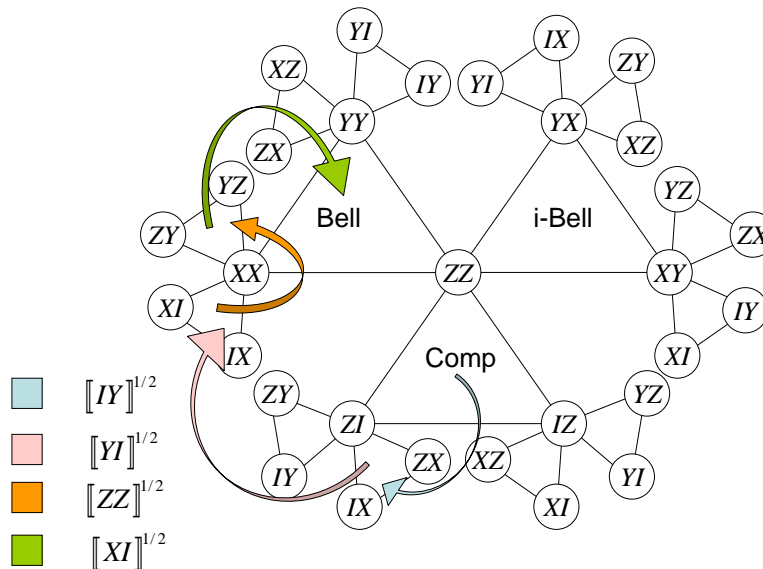
$$[[ZZ]]^{1/2} YY = YY$$

$$\{IY, YI, YY\} \xrightarrow{[[ZZ]]^{1/2}} \{ZX, XZ, YY\}$$

signs unimportant in basis representation

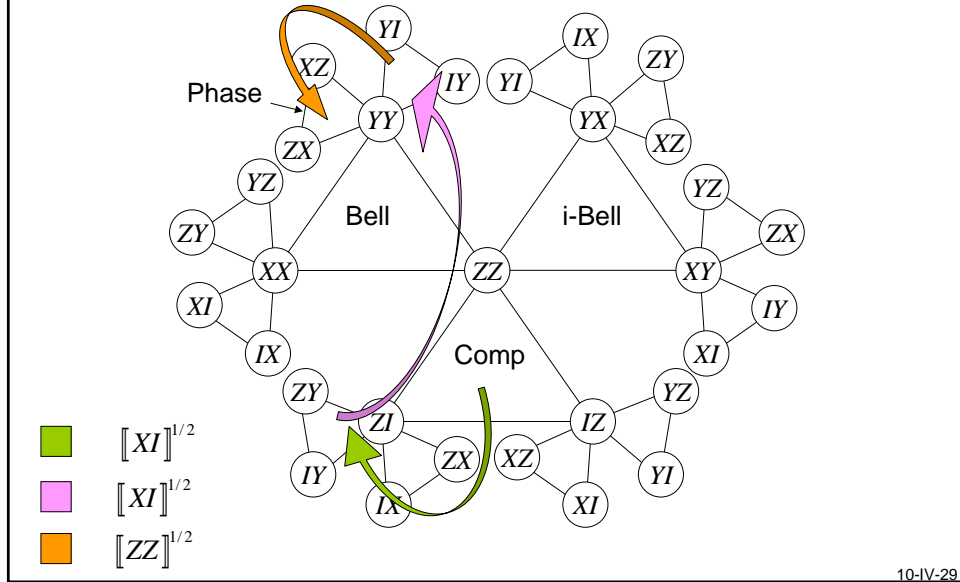
10-IV-27a

VISUALIZING THE CLIFFORD MOVES ON THE STABILIZER MAP (1)



10-IV-28

VISUALIZING THE CLIFFORD MOVES ON THE STABILIZER MAP (2)



END OF LECTURE