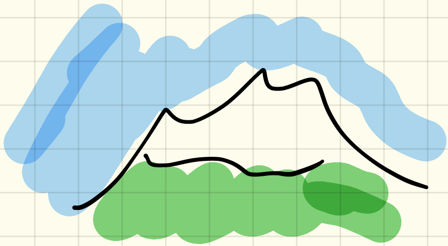
Bosonic Gottesman-Kitaev-Preskill Code [GKP]

1. GKP code, modular variables
2. Errors and gates
3. Repeated QEC & decoding
4. Perspective
5. Some references



Lecture Notes
Les Houches 2019

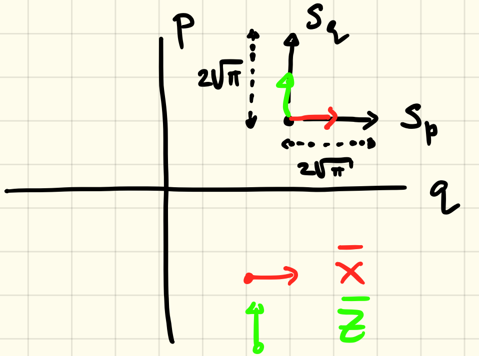
②

Single oscillator

$$[a, a^\dagger] = I \qquad \hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger)\ ; \ \hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a)$$

dimensionless quadratures

$$[\hat{q}, \hat{p}] = iI$$



$$S_p = e^{-i 2\sqrt{\pi}\,\hat{p}} = D(\sqrt{2\pi})$$

$$S_q = e^{i 2\sqrt{\pi}\,\hat{q}}$$

Commute!

$$\left(e^A e^B = e^B e^A e^{[A,B]} \quad \text{when } [A,B] \propto I \right)$$

$$\exp(-iv\hat{q})|p\rangle = |p - v\rangle$$

$$\exp(-iu\hat{p})|q\rangle = |q + u\rangle$$

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

e.g. $\hat{q}$ = position $\times \sqrt{\frac{\text{mass} \times \omega}{\hbar}}$

$\hat{p}$ = momentum $/ \sqrt{\frac{\text{mass} \times \omega \times}{\hbar}}$

for mechanical oscillator

2 extra

$$e^A e^B = e^{A+B+[A,B]/2+\cdots}$$

$$e^B e^A = e^{A+B-[A,B]/2+\cdots}$$

$$e^{A+B} = e^A e^B e^{-[A,B]/2} \implies e^A e^B = e^B e^A e^{[A,B]}$$

$$e^{A+B} = e^B e^A e^{[A,B]/2}$$

$\bar{X} = e^{-i\sqrt{\pi}\,\hat{p}}$, $\bar{Z} = e^{i\sqrt{\pi}\,\hat{q}}$ $\qquad \bar{X}\bar{Z} = -\bar{Z}\bar{X}$

note $\bar{X} \neq \bar{X}^\dagger$, but $\bar{X} = \bar{X}^\dagger S_p$ and on codespace $S_p = +1$

Measure eigenvalue of S_p ?

= measure $p \bmod \sqrt{\pi} = a + k\sqrt{\pi} \quad k \in \mathbb{Z}$

As S_q commutes, one can simultaneously measure $q \bmod \sqrt{\pi}$

$\rightsquigarrow$ useful as sensor for small changes in p and q ?

GKP code :

Choose common +1 eigenspace of S_p and S_q (other eigenspace will also do)

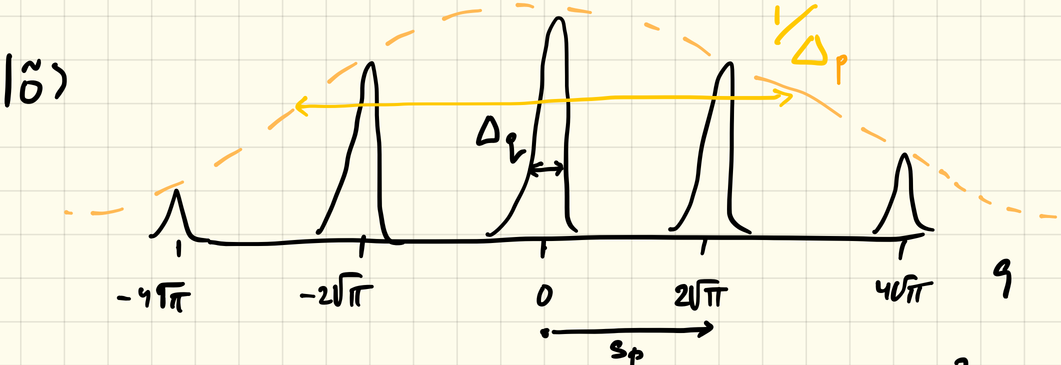
as code space.

$\Rightarrow \quad p \bmod \sqrt{\pi} = q \bmod \sqrt{\pi} = 0$

$|0\rangle$ has $q = 2k\sqrt{\pi}$ $k \in \mathbb{Z}$.

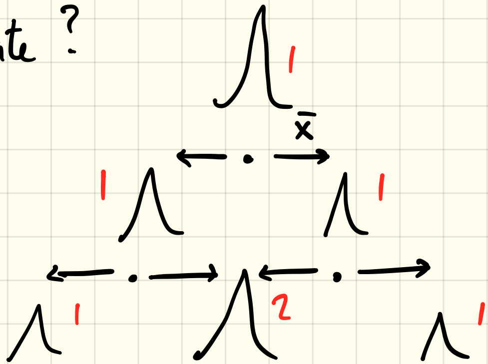
$|1\rangle$ has $q = (2k+1)\sqrt{\pi}$

Characterization of finite $\bar{n}$ states using squeezing parameters $\Delta_p, \Delta_q < 1$



Single peak at center: squeezed vacuum state $\Delta^2(q) = \Delta_q^2 = (q - \langle q \rangle)^2$

Create this state?



Binonomial sum of displacement after M rounds $\rightsquigarrow$ Gaussian envelope
$\binom{M}{k}$

5

More formally

$$|\tilde{0}\rangle = \underbrace{\frac{1}{\sqrt{\pi\Delta^2}} \int d^2\alpha \; \exp(-|\alpha|^2/\Delta^2)\, D(\alpha)}_{=E} \; |\bar{0}\rangle \qquad \Delta p = \Delta q = \Delta$$

Gaussian shift errors on top of perfect state

One can evaluate $\langle \tilde{0} | \bar{n} | \tilde{0} \rangle \approx \frac{1}{2\Delta^2} - \frac{1}{2}$

$\bar{n} = 4 \rightarrow \Delta = 1/3 \qquad (\Delta = 0.2 \equiv 8.3\ \mathrm{dB})$

Quality of state ?

Use e.g. $\Delta^2_{p/q}(\rho) = \frac{1}{2\pi}\left[\underbrace{\frac{1}{|\mathrm{Tr}\, S_{p/q}\,\rho|^2}}_{\text{Holevo phase variance}} - 1\right]$

$$\langle e^{i\theta} \rangle = \int_{-\pi}^{\pi} d\theta \; P(\theta)\, e^{i\theta} = \mathrm{Tr}\, S_{p/q}\, \rho$$

Using Gaussian integrals one can find that

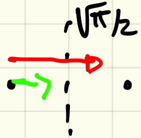
$E = 2\sqrt{\pi\Delta^2}\, \exp(-\Delta^2 \hat{n})$

Writing out $|\tilde{0}\rangle$ in terms of $|\bar{0}\rangle \propto \sum_{n\in\mathbb{Z}} \delta(q-2n\sqrt{\pi})|q\rangle$ 5 extra

performing Gaussian integral over α:

$$|\tilde{0}\rangle \propto \int dq \sum_{n\in\mathbb{Z}} e^{-2\Delta^2\pi n^2} \, e^{-\frac{1}{2\Delta^2}(q-2n\sqrt{\pi})^2} |q\rangle$$

envelope squeezed peaks



2. Errors and Gates

$\downarrow$ small shifts $e^{i\varepsilon\hat{p}}$, $e^{i\varepsilon\hat{q}}$ change eigenvalues of S_p and S_q.

$\rightarrow$ detect errors

$|\varepsilon| < \sqrt{\pi}/2$ "less than half a logical" are correctable

apply smallest shift such that S_p & S_q have eigenvalue $\mathbb{I}$.

$$S_q\, e^{i\varepsilon\hat{p}}|\bar{\psi}\rangle = e^{i2\sqrt{\pi}\hat{q}}\, e^{i\varepsilon\hat{p}}|\bar{\psi}\rangle = e^{i\varepsilon\hat{p}}\, e^{i2\sqrt{\pi}\hat{q}}\, e^{-2\sqrt{\pi}\varepsilon i}|\bar{\psi}\rangle$$

new eigenvalue

"Real" errors: photon loss, $K(a^\dagger a)^2$ terms, finite-$\bar{n}$ codewords, errors in gates ...

Why correctable at small rate?

subtle, depends on $\bar{n}$ of state that you apply it to...

$\gamma = \varkappa t$

extra 7

Photon Loss Example

$E_0 = I - \frac{\gamma}{2} a^\dagger a/2$; $E_1 = \sqrt{\gamma}\, a$ small $\gamma \bar{n} << 1$

as Kraus operators.

Write expansion for $\sqrt{\gamma}\, a$ in terms of small shifts, e.g.

$\sqrt{\gamma}\, a = x_1 + i x_2$ $\quad x_1 = \sqrt{\frac{\gamma}{2}}\, p$; $x_2 = \sqrt{\frac{\gamma}{2}}\, q$

Use $\arcsin(x) = x + \frac{x^3}{6} + \ldots$ (small x)

$x_1 = \arcsin(\sin x_1)$
$x_2 = \arcsin(\sin x_2)$

so $\sqrt{\gamma}\, a = \sin\left(p\sqrt{\frac{\gamma}{2}}\right) + i \sin\left(q\sqrt{\frac{\gamma}{2}}\right) + \frac{\sin^3\left(p\sqrt{\frac{\gamma}{2}}\right)}{6} + i\, \frac{\sin^3\left(q\sqrt{\frac{\gamma}{2}}\right)}{6}. + \ldots$

= linear combination of small shifts $\varepsilon = \sqrt{\frac{\gamma}{2}}$

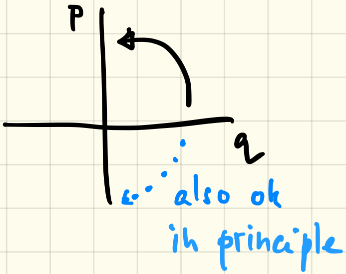
Before we discuss how to prepare or correct GKP states, let's talk about gates first ...

Hadamard H, CNOT, $\bar{X}$ and $\bar{Z}$ measurement, T gate $= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

$\bar{X} \longleftrightarrow \bar{Z}$

$= \exp\left(i \frac{\pi}{2} a^\dagger a\right)$

"just wait"



"homodyne" measurement of p and determine whether it is closest to even or odd multiple of $\sqrt{\pi}$.

See next slides, not used in error correction

$p \to -q$
$q \to p$
$\Rightarrow \quad e^{i\sqrt{\pi} p} \longrightarrow e^{-i\sqrt{\pi} q} = \bar{Z}^\dagger = \bar{Z}$

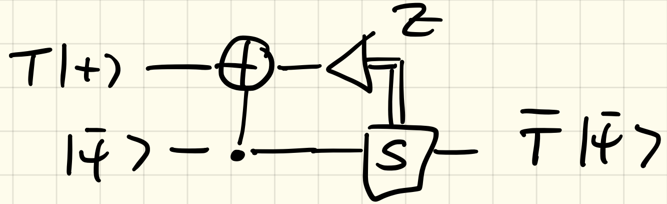
only on perfect codespace

" $\hat{p} \to X$ "
" $\hat{q} \to Z$ "

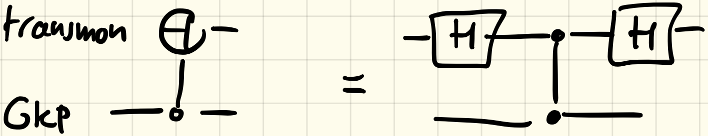
8 extra

T-gate in 2 ways.

1.



$T|+\rangle$ could be GKP codeword or another qubit (say transmon). In latter case we need



CZ is a qubit-controlled displacement.

2. Make a +1 eigenstate of Hadamard $= \exp(i \frac{\pi}{2} a^\dagger a) =$ codeword with $n = 0 \bmod 4$: Start with $|\text{vac.}\rangle$ and measure S_p & S_q, post-select on $S_p = S_q = +1$

$-\boxed{S}-$ gate $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$S|0\rangle = |0\rangle$

$S^\dagger X S = -Y$

$S^\dagger Z S = Z$

$q \to q$

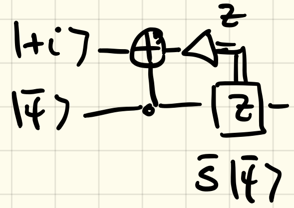
$p \to p - q$

Determine what interaction this requires...

(exercise)

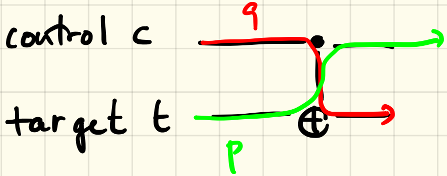
⑧ extra-extra

Alternative:



"$p \to X$
$q \to Z$"

CNOT :



X error = a change in q propagates from control to target.

$$q_t \rightarrow q_c + q_t \qquad CNOT^{\dagger}\, q_t\, CNOT = q_c + q_t$$

$$p_c \rightarrow p_c - p_t \qquad CNOT^{\dagger}\, p_c\, CNOT = p_c - p_t$$

$$q_c \rightarrow q_c \qquad \Rightarrow \quad CNOT = \exp(-i\, \hat{p}_t\, \hat{q}_c)$$

$$p_t \rightarrow p_t$$

How to get this interaction, between 2 cavity modes in circuit-QED

$$H_t^{\dagger}\, CNOT\, H_t = \exp(-i\, \hat{q}_t\, \hat{q}_c)$$

" $\hat{p} \rightarrow X$

$\hat{q} \rightarrow Z$ "

9 extra

3-wave mixing interaction

$$\sim (a_t + a_t^\dagger)(b_c + b_c^\dagger)(c_p + c_p^\dagger)$$

$\downarrow$ target $\quad \downarrow$ control $\quad \downarrow$ pump

in rotating frame of target and control oscillator

$a_t \to a_t \, e^{-i\omega_t t}$ etc. , gives :

$$\left(a_t b_c \, e^{-i(\omega_t + \omega_c)t} + h.c.\right)(c_p + c_p^\dagger)$$

$$\left(a_t b_c^\dagger \, e^{-i(\omega_t - \omega_c)t} + h.c\right)(c_p + c_p^\dagger)$$

Two-tone pump with equal amplitudes $\langle c_p \rangle = \alpha \left[e^{-i(\omega_t + \omega_c)t} + e^{-i(\omega_t - \omega_c)t} \right]$

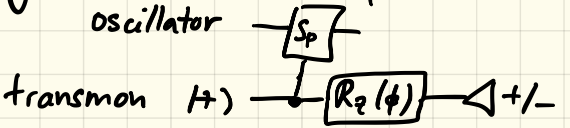
at $\omega_p = \omega_t + \omega_c$

and $\omega_p = \omega_t - \omega_c$

gives $\approx \alpha \left[a_t b_c + a_t^\dagger b_c^\dagger + a_t b_c^\dagger + a_t^\dagger b_c \right] = 2\alpha \, \hat{q}_c \hat{q}_t$

State Preparation and Quantum Error Correction

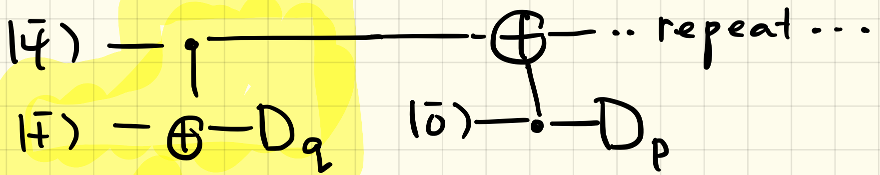
- measure eigenvalues of S_p and S_q using transmon ancilla qubits



qubit-controlled displacement

- Steane QEC using an ancilla GKP oscillator $\longrightarrow$ more fault-tolerant

QEC unit:

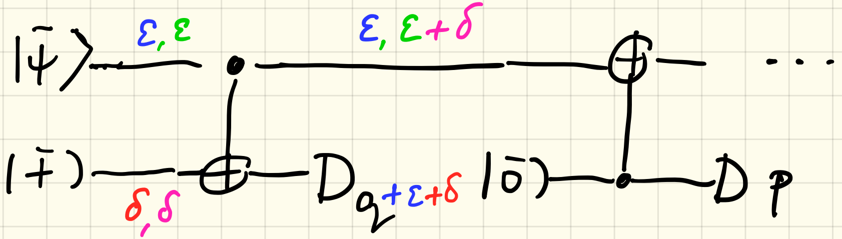


q-type errors

$= e^{i\varepsilon\hat p}$ = corrected by mod. meas. of S_q

$\hat p \to X$

$\hat q \to Z$



Consider error $e^{ip\varepsilon}$ on data qubit $(e^{iq\varepsilon})$

and $e^{ip\delta}$ on ancilla qubit $(e^{iq\delta})$

δ is feedback error.

Repeat QEC unit → data $q_1, q_2 \cdots q_M$ & $p_1, p_2 \cdots p_M$.

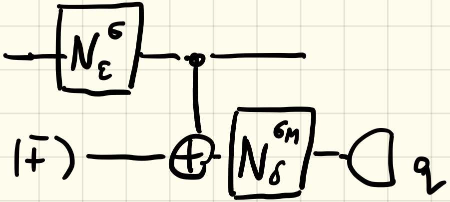
Decoding ? (= theory building)

- Strategy 0 : ignore δ's, trust measurement
- Define optimal, max. likelihood decoder and approximate it by "minimum energy decoder"

3. Decoding

First simplify $\begin{cases} \bullet \text{ only correcting } q\text{-type errors} \\ \bullet \text{ stochastic error model instead} \end{cases}$

of coherent superposition of shifts (remember $|\tilde{0}\rangle$)



rescale $P = 2\sqrt{\pi}\, p$
$Q = 2\sqrt{\pi}\, q$
logical $= \sqrt{\pi} \to 2\pi$

for convenience

N_ε^σ does $Q \to Q + \varepsilon$ with Gaussian Probability $P_\sigma(\varepsilon)$ (Similarly $N_\delta^{\sigma_M}$)

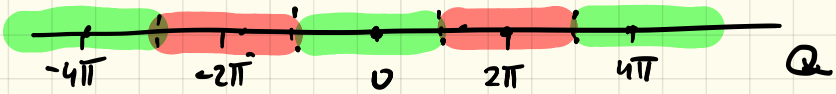
Cumulative shift after t rounds on data oscillator:

$$\phi_t = \varepsilon_1 + \varepsilon_2 \ldots + \varepsilon_t \qquad \phi_0 = 0$$

and outcome at round t $= q_t = \phi_t + \delta_t \mod 2\pi$

After last round, we imagine measuring (perfectly) $\hat{q}$ on the data oscillator and deciding whether q is closer to an even or odd multiple of $\sqrt{\pi}$.

The measurement record $q_1, \dots q_M$ should let us decide whether to flip this conclusion or not, i.e whether the cumulative shift ϕ_M is close to an even or odd multiple of 2π:



$I_0 =$ intervals

$I_1 =$ intervals

Remember rescaled variables
2π = logical shift
4π = stabilizer shift

What is probability that $\phi_M \in I_0$?

$$\mathrm{Prob}(\phi_M \in I_0 \mid \vec{q}\,) = \int_{I_0} d\phi_1\, d\phi_2 \ldots \int d\phi_M \, \mathrm{Prob}(\vec{\phi} \mid \vec{q}\,)$$

[compare with $\mathrm{Prob}(\phi_M \in I_1 \mid \vec{q}\,)$]

$$= \int_{I_{0/1}} D\vec{\phi} \; \frac{\mathrm{Prob}(\vec{q} \mid \vec{\phi}\,)\, P(\vec{\phi}\,)}{P(\vec{q}\,)}$$

drops out in comparison.

$$= \int_{I_{0/1}} D\vec{\phi} \; \exp\left(-H_{\vec{q}}(\vec{\phi}\,)\right)$$

hamiltonian with randomness set by $\vec{q}$.

$$H_{\vec{q}}(\vec{\phi}\,) \approx \sum_{t=1}^{M} \frac{(\phi_t - \phi_{t-1})^2}{2\sigma^2} + \frac{1}{\sigma_M^2} \sum_{t=1}^{M} \cos(q_t - \phi_t)$$

error term "kinetic energy"

measurement term "potential energy"

Perspective

- integrate GKP code into an architecture

  : surface-GKP code $\longrightarrow$ $\bar{n} \approx 4$ threshold

- effect of coherent errors, new decoding questions

- is GKP qubit a better qubit?

  CNOT and quality of state prep. will be crucial

References:

- Gottesman, Kitaev, Preskill: "Encoding a qubit into an oscillator", Phys. Rev. A 64, 012310 (2001)
- Terhal & Weigand: "Encoding a qubit into a cavity mode in circuit-QED using phase estimation", Phys. Rev. A 93, 012315 (2016)
- V. Albert et al.: "Performance and structure of single-mode bosonic codes", Phys. Rev. A, 032346 (2018)
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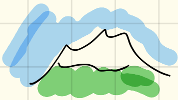

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