Ultrastrong coupling circuit QED: vacuum degeneracy and quantum phase transitions

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Acknowledgments for the work presented today

**Pierre Nataf** (Ph.D. student) [*Circuit QED*]

S. De Liberato (PhD 2009) [*Cavity QED in semiconductors*]

D. Hagenmüller (PhD student) [*Cavity QED in semiconductors*]

I. Carusotto (University of Trento, Italy)

…. (mentioned during the talk)
Outline

- Introduction on cavity and circuit QED
- Ultrastrong coupling regime
- Quantum phase transitions and vacuum degeneracy in circuit QED
- Conclusions and perspectives
Cavity quantum electrodynamics (cavity QED) is the study of the interaction between light confined in a reflective cavity and atoms or other particles, under conditions where the quantum nature of light photons is significant.

See for example:
S. Haroche, J.-M. Raimond,

Examples of solid-state cavity QED systems

• Quantum dots or quantum wells in dielectric cavities (semiconductor cavity QED)

See for example:
- …

• Josephson junction artificial atoms in transmission line resonators (superconducting circuit QED)

See for example:
- M.H. Devoret, Lectures at Collège de France (years 2008, 2009)
The simplest cavity QED system: a 2-level atom + a photon mode

Two-level system
(fermionic system, Anharmonic spectrum)

Cavity photon Fock space
(Bosonic field, harmonic oscillator)
Jaynes-Cummings Hamiltonian and conserved number

\[ H_0 = \text{const.} + \hbar \omega_e |e\rangle \langle e| + \hbar \omega_g |g\rangle \langle g| + \hbar \omega_{cav} a^\dagger a \]

\[ H_{int} = \hbar \Omega_0 \left( |e\rangle \langle g| a + a^\dagger |g\rangle \langle e| \right) \]

\[ \hat{N}_{exc} = a^\dagger a + |e\rangle \langle e| \]

Total excitation number (cavity + atom)

\[ [\hat{N}_{exc}, H_0 + H_{int}] = 0 \]

Conserved by JC Hamiltonian!
Vacuum Rabi splitting in the frequency domain

Number (Fock) states

\[ |n\rangle = \frac{a_+^n}{\sqrt{n!}} |0\rangle \]

Eigenstates

\[
\begin{align*}
|e\rangle & \quad \text{Resonant case} \\
|g\rangle & \\
|0\rangle & \\
|1\rangle & \\
|2\rangle & \\
|3\rangle & \\
|\ldots\rangle & \\
\end{align*}
\]

Energies:

\[ H|\pm, n\rangle = (n\hbar \omega \pm \hbar \Omega_0 \sqrt{n+1})|\pm, n\rangle \]

\[
|\pm, n\rangle = \frac{1}{\sqrt{2}} (|e, n\rangle \pm |g, n+1\rangle)
\]
What about the vacuum (ground state)? Unchanged

The vacuum is NOT changed by light-matter interaction in the Jaynes-Cummings Model!

\[ |g,0\rangle = |g\rangle \otimes |0\rangle \]
Vacuum Rabi coupling (electric dipole case)

- Reversible exchange of energy between the atom and the photon field.

- Coupling quantified by vacuum Rabi frequency

\[ \hbar \Omega_0 = dE_0 \]

Electric dipole \hspace{2cm} Electric field vacuum fluctuations
Vacuum Rabi frequency: back-of-the-envelope calculation

\[ \hbar \Omega_0 = dE_0 \]

Electric dipole of two-level transitions

\[ d = e \int dV \phi_e^*(r) r \phi_g(r) \]

Energy of one photon

\[ \hbar \omega \sim \epsilon_0 \int E^2 dV \sim \epsilon_0 E_0^2 V \]

Atomic size

\[ d \sim e L_{at} \]

Vacuum electric field

\[ E_0 \sim \sqrt{\frac{\hbar \omega}{\epsilon_0 V}} \]
Strong coupling regime

Vacuum Rabi coupling larger than photon and atom losses

\[ \Omega_0 > \gamma, \kappa \]

Recipes for strong coupling:
- very small losses OR/AND
- very large coupling
Electrical dipole coupling: limit imposed by fine structure constant

\[ \frac{\Omega_0}{\omega} \sim \sqrt{\frac{d^2}{\epsilon_0 V \hbar \omega}} \]

For atoms in a \( \lambda^3 \) cavity:

\[ \frac{\Omega_0}{\omega} \sim \sqrt{\frac{e^2 L_{at}^2}{\epsilon_0 \hbar c \lambda^2}} = \frac{L_{at}}{\lambda} \sqrt{4\pi \sqrt{\alpha}} \ll \sqrt{\alpha} \]

In the case of Rydberg atoms in a microwave cavity:

\[ \frac{\Omega_0}{\omega} \sim 10^{-6} \] (for a single atom)

Ultrastrong coupling regime

\[ \frac{\Omega_0}{\omega_{eg}} \gg 1 \]

- **Weak coupling**
  - Loss rate
- **Strong coupling**
  - Transition Frequency
- **Ultrastrong coupling**
  - \( \gamma \)
  - \( \omega_{eg} \)

0 \[ \Omega_0 \]
Collective vacuum Rabi coupling

\[ N \] identical atoms coupled to the same photon mode

\[ \Omega_0 \propto \sqrt{N} \]

- Vacuum Rabi frequency is enhanced by collective excitation
- Collective excitations are bosonic for \( N \gg 1 \)
- In principle, anharmonicity is lost …
How to reach ultrastrong coupling in semiconductors

Ultrastrong coupling due to collective vacuum Rabi coupling in **semiconductors**:

- **2D electron gas in semiconductor microcavities**
  CC, G. Bastard, I. Carusotto, PRB 72, 115303 (2005)

- **2D electron gas with magnetic field**
  D. Hagenmüller, S. De Liberato, CC, PRB 81, 235303 (2010)
State-of-the-art in semiconductors


Y. Todorov, … C. Sirtori, unpublished (Lab MPQ, Univ. Paris Diderot)
How to reach the ultrastrong coupling in superconductors

- Inductive coupling of a Josephson atom to a transmission line resonator:


- Collective vacuum Rabi coupling of Josephson chains:

What happens in ultrastrong coupling cavity QED?

*More is different!*

- Quantum vacuum (ground state) depends on interaction strength

- Antiresonant (non-rotating wave) terms of light-matter interaction becomes important
What are the antiresonant terms?

\[ H_{anti} = \hbar \Omega_0 \left( |e\rangle \langle g| a^\dagger + a |g\rangle \langle e| \right) \]

NOTE: Antiresonant (non-rotating wave) terms are neglected in the Jaynes-Cummings model.

Destruction of two excitations

Creation of two excitations

\[ |e\rangle \langle g| a^\dagger \]

ANTIRESONANT coupling
The standard vacuum $|g, 0\rangle$ is no longer the quantum ground state!
Total excitation number is no longer conserved

\[ \hat{N}_{exc} = a^\dagger a + |e\rangle\langle e| \]

\[ [\hat{N}_{exc}, H_{anti}] \neq 0 \]

\[ N_{exc} = 0 \quad \text{and} \quad N_{exc} = 2 \]

|g, 0\rangle \quad \text{ANTIRESONANT coupling} \quad |e, 1\rangle
Parity is conserved

Parity operator

\[ \hat{P}_{exc} = \exp(i\pi \hat{N}_{exc}) \]

\[ [\hat{P}_{exc}, H_{anti}] = 0 \]

\[ N_{exc} = 0 \]

\[ \begin{align*}
|e\rangle & \quad |g\rangle \\
\vdots & \quad |1\rangle \\
|2\rangle & \quad |0\rangle
\end{align*} \]

ANTIRESONANT coupling

\[ N_{exc} = 2 \]

\[ \begin{align*}
|e\rangle & \quad |g\rangle \\
\vdots & \quad |1\rangle \\
|2\rangle & \quad |0\rangle
\end{align*} \]
The form of the ground state with finite antiresonant interactions

\[ |G\rangle = ...|g, 0\rangle + ...|e, 1\rangle + ...|g, 2\rangle + ...|e, 3\rangle + ...|g, 4\rangle + ... \]

The ground state contains photons \(! \langle G | a^\dagger a | G \rangle > 0\!\)

The total number of excitations (matter + photon) is even \( \langle G | \hat{P}_{exc} | G \rangle = 1 \)
(unless a symmetry breaking)
When are the antiresonant terms negligible?

Antiresonant terms negligible only if \( \frac{\Omega_0}{\omega} \ll 1 \)

Perturbative theory argument:

\[ \langle e, 1 | H_0 | e, 1 \rangle - \langle g, 0 | H_0 | g, 0 \rangle \approx 2\hbar \omega \]

Difference between bare energies

\[ \langle g, 0 | H_{anti} | e, 1 \rangle = \hbar \Omega_0 \]

Coupling energy

\[ H_{anti} = \hbar \Omega_0 \left( |e\rangle \langle g| a\dagger + a |g\rangle \langle e| \right) \]
The ‘virtuality’ of photons in the ground state (vacuum)

\[ |G\rangle = \ldots |g, 0\rangle + \ldots |e, 1\rangle + \ldots |g, 2\rangle + \ldots |e, 3\rangle + \ldots |g, 4\rangle + \ldots \]

The photons in the ground state can NOT escape the cavity!

The ground state is the lowest energy state!
Non-adiabatic release of quantum vacuum photons

External Control Bias

Time-dependent vacuum Rabi frequency
\[ \Omega_0(t) = \bar{\Omega} + \Omega_0^{mod}(t) \]

A *gedanken* experiment: Sudden switch-off

Anomalous vacuum
\[ |G\rangle \]

Non-adiabatic dynamics

Standard vacuum
\[ |g, 0\rangle \]

Quantum vacuum radiation

\( \tau_{relax} \)

TIME \( t \)

CC, G. Bastard, I. Carusotto, PRB 72, 115303 (2005)
Quantum vacuum radiation by modulating vacuum Rabi coupling

\[ \Omega_0(t) = g_0 + \Delta g \sin(\omega_{mod}t) \]

- For a dissipative \textit{two-level} (qubit) system:
- For a dissipative \textit{bosonic} (polaritons) system:
Emerging field: non-adiabatic cavity QED

Sub-cycle switch-on of ultrastrong light–matter interaction

G. Günter¹, A. A. Anappara¹,², J. Hees¹, A. Sell¹, G. Biasiol³, L. Sorba²,³, S. De Liberato⁴,⁵, C. Ciuti⁴, A. Tredicucci², A. Leitenstorfer¹ & R. Huber¹
Problems in ultrastrong coupling circuit QED

- What happens when $N$ artificial atom are embedded in a transmission line resonator?? Quantum phase transitions??

- What happens with different types of coupling?

- Differences/analogies with cavity QED?

N Josephson atoms inductively coupled to a resonator

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The strength of inductive coupling in circuit QED


Circuit-QED: How strong can the coupling between a Josephson junction atom and a transmission line resonator be?

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After reviewing the limitation by the fine structure constant α of the dimensionless coupling constant of an hydrogenic atom with a mode of the electromagnetic field in a cavity, we show that the situation presents itself differently for an artificial Josephson atom coupled to a transmission line resonator. Whereas the coupling constant for the case where such an atom is placed inside the dielectric of the resonator is proportional to $\alpha^{1/2}$, the coupling of the Josephson atom when it is placed in series with the conducting elements of the resonator is proportional to $\alpha^{-1/2}$ and can reach values greater than 1.

Giant coupling: $\frac{\Omega_0}{\omega} \gg 1$ even with a single Josephson atom !!
Circuit quantum Hamiltonian

\[ H = H_{res} + H_F + H_{coupling} \]

Resonator part

\[ H_{res} = \sum_{j=1}^{N} 4E_{Cr}(\hat{N}_r^j)^2 + E_{Lr}\frac{(\hat{\phi}_r^j - \hat{\phi}_r^{j-1})^2}{2}, \]

Josephson atomic part

\[ H_F = \sum_{j=1}^{N} 4E_{CJ}(\hat{N}_J^j)^2 + E_{LJ}\frac{(\hat{\phi}_J^j)^2}{2} - E_J \cos(\hat{\phi}_J^j + \frac{2e}{\hbar}\Phi_{ext}) \]

Inductive coupling between Resonator and Josephson atoms

\[ H_{coupling} = \sum_{j=1}^{N} G(\hat{\phi}_r^j - \hat{\phi}_r^{j-1})\hat{\phi}_J^j, \]
Artificial atom Hamiltonian in terms of pseudospins

Site-dependent Pauli matrices

\[ \hat{\sigma}_{x,j} = \hat{\sigma}_{+,j} + \hat{\sigma}_{+,j}^\dagger \]
\[ \hat{\sigma}_{y,j} = -i(\hat{\sigma}_{+,j} - \hat{\sigma}_{+,j}^\dagger) \]
\[ \hat{\sigma}_{z,j} = 2\hat{\sigma}_{+,j}\hat{\sigma}_{+,j}^\dagger - 1 \]

2-level system

\[ \hat{\sigma}_{+,j} = |g\rangle \langle e|_j \]

Artificial atom bare energy:

\[ H_{F,J} \simeq \hbar \omega_F \frac{1}{2} \hat{\sigma}_{z,j} \]

Josephson atom flux field

\[ \dot{\phi}_J \simeq -\varphi_{01}\hat{\sigma}_{x,j} \]

 Fluxonium* atom

Mode expansion for transmission line resonator

Resonator flux field operator

\[ \hat{\phi}(x) = i \sum_{k \geq 1} \frac{1}{\omega_k} \sqrt{\frac{\hbar \omega_k}{2c_r}} f_k(x) (\hat{a}_k - \hat{a}_k^\dagger) \]

Mode frequencies

\[ \omega_k = \frac{k \pi}{d} \frac{1}{\sqrt{l_r c_r}} \text{ with } k = 1, 2, 3, \ldots \]

\[ f_k(x) = -\sqrt{2/d} \sin\left(\frac{k\pi x}{d}\right) \text{ for } k \text{ odd} \]

\[ f_k(x) = \sqrt{2/d} \cos\left(\frac{k\pi x}{d}\right) \text{ for } k \text{ even} \]

For the resonator quantization, see, e.g., A. Blais, R-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, PRA 69, 062320 (2004))
Multimode spin-boson Hamiltonian:

\[ H \sim constant + \sum_{k=1\ldots N_m} \hbar \omega_k a_k a_k^\dagger + \sum_j \frac{\hbar \omega_F}{2} \hat{\sigma}_{z,j} + \sum_{k=1\ldots N_m} \sum_{j=1}^N i\Omega_k \sqrt{\frac{2}{N}} \Delta f_k(x_j) (a_k - a_k^\dagger) \hat{\sigma}_{x,j} \]

Note: reminiscent of Dicke model Hamiltonian

Note: Analogous to the MAGNETIC COUPLING of real Spins to a cavity field

\[ H_{int} = \sum_j \vec{I}_j \cdot \vec{B}_{cav} \]
Normalized vacuum Rabi frequency

\[ \frac{\Omega_{k=1}}{\omega_{k=1}} = g\sqrt{N} = \sqrt{\frac{Z_{\text{vac}}}{2Z_r\alpha}}\mu\nu\chi\sqrt{N} \sim 5.7\chi\sqrt{N} \]

Branching ratio (it allows to tune the coupling)

\[ \chi = \left( \frac{L_r}{L_1L_r + L_1L_2 + L_2L_r} \right)^{\frac{1}{4}} \frac{L_1}{(L_1 + L_2)^{\frac{3}{4}}} \]

0 ≤ \chi ≤ 1

Parameters and constants:

\[ Z_r = \sqrt{\frac{L_r}{C_r}} = 50\Omega \]

\[ \frac{Z_{\text{vac}}}{2\alpha} = \frac{h}{e^2} = R_k \sim 25.8k\Omega \]

\[ \mu = \sin\left(\frac{\pi\alpha}{2d}\right) \]

\[ \nu = \frac{1}{4\pi}\varphi_{01} \sim \frac{1}{4} \]
Energy spectrum for a finite-size system

Degenerate ground state

5 artificial Josephson atoms

2 Degenerate vacua in the ultrastrong coupling limit

\[ |G_+\rangle = C_G \Pi_j |+\rangle_j \otimes \Pi_{k_o} e^{\left(\frac{-\alpha \sqrt{2}}{k_o^3} \sin \left(\frac{\pi}{2N}\right) a_{k_o}^\dagger\right)} |0\rangle_{k_o} \otimes \Pi_{k_e} |0\rangle_{k_e} \]

\[ |G_-\rangle = C_G \Pi_j |-\rangle_j \otimes \Pi_{k_o} e^{-\left(\frac{\alpha \sqrt{2}}{k_o^3} \sin \left(\frac{\pi}{2N}\right) a_{k_o}^\dagger\right)} |0\rangle_{k_o} \otimes \Pi_{k_e} |0\rangle_{k_e} \]

\[ \hat{\sigma}_{j,x} |\pm\rangle_j = \pm |\pm\rangle_j \]

« Ferromagnetic » state for Josephson atoms

Coherent state for resonator field
Degeneracy splitting for finite coupling/size

Finite-size scaling (analytically calculated)

\[ \delta = \omega_F \exp(-g^2 \beta(N)) \]

- Frequency splitting
- Josephson Atom transition frequency
- \( g \) = Normalized Vacuum Rabi coupling (per atom)

Degeneracy splitting: numerically calculated finite-size scaling

\[ \delta = \omega_F \exp(-g^2 \beta(N)) \]

\[ \beta(N) \sim 2N^2 \]
Protection with respect to some kind of local noise sources

Degeneracy protected with respect to the following kind of perturbation:

\[ H_{\text{pert}} = \sum_j \frac{\hbar \Delta_j}{2} \hat{\sigma}_{z,j} + \frac{\hbar \Delta_j}{2} \hat{\sigma}_{y,j} \]

No effect up to N-th order perturbation theory:

\[ \langle G_{\pm}|H^m_{\text{pert}}|G_{\pm} \rangle = \langle G_{\pm}|H^m_{\text{pert}}|G_{\mp} \rangle = 0 \text{ with } m \leq N - 1 \]

(Partial) analogy with protected systems:
Effect of noise: numerical study

\[ \omega_{j,F} = \omega_F + \Delta_j = \omega_F(1 + 0.5\xi_j) \]

Average splitting

Still Exponential!
Degenerate vacua as a valuable qubit

- Encode quantum information in degenerate vacuum subspace

\[ |\Psi\rangle = c_+ |G_+\rangle + c_- |G_-\rangle \]

- Protection with respect to some kind of noise

- In the ultrastrong coupling limit, insensitivity to variation of Josephson elements

- Unprotected channel can be used to perform operations and readout
Quantum phase transition in the ‘thermodynamic’ limit

\[ N \gg 1 \text{ (thermodynamical limit)} \]

\[ P_{exc} = +1 \]

\[ |G\rangle \]

\[ G_+ \] \quad \[ G_- \]

‘Normal’ vacuum \quad Degenerate vacua

\[ \Omega_0^{crit} \]

Critical vacuum Rabi frequency:

\[ \Omega_0^{crit} = \frac{1}{2} \sqrt{\omega_{eg}\omega_{cav}} \]
Energy of elementary bosonic excitations for $N >> 1$

Possible scenarios …

Dicke Model
\(D = 0\)

\[\frac{\omega}{\omega_{eg}}\]

\(\Omega_0/\omega_{eg}\)

\(0\) to \(2\)

Hopfield Model
\(D = \Omega_0^2/\omega_{eg}\)

\[\frac{\omega}{\omega_{eg}}\]

\(\Omega_0/\omega_{eg}\)

\(0\) to \(2\)

\(0 < D < \Omega_0^2/\omega_{eg}\)

\(\frac{\omega}{\omega_{eg}}\)

\(\Omega_0/\omega_{eg}\)

\(0\) to \(2\)

\(D > \Omega_0^2/\omega_{eg}\)

\(\frac{\omega}{\omega_{eg}}\)

\(\Omega_0/\omega_{eg}\)

\(0\) to \(2\)

Shifted critical point

No critical point

No critical point

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P. Nataf, CC, submitted; preprint arXiv:1006.1801
Electric dipole coupling in cavity QED

\[ H = \sum_j \frac{1}{2m} \left( p_j - q_j \hat{\mathbf{A}}(r_j) \right)^2 + V_j + H_{cav} \]

Electrical dipole approximation
\[ \hat{\mathbf{A}}(r_j) \simeq \hat{\mathbf{A}}_0 \]

\[ H_{int} = -\sum_j \frac{q_j}{m} p_j \cdot \hat{\mathbf{A}}_0 \rightarrow \Omega_0 (a + a^\dagger) \frac{1}{\sqrt{N}} \sum_j \hat{\sigma}_{x,j} = \Omega_0 (a + a^\dagger)(b + b^\dagger) \]

Bosonic excitation

\[ H_{A^2} = \sum_j \frac{q_j^2}{2m} \hat{\mathbf{A}}_0^2 \rightarrow D (a + a^\dagger)^2 \]
The value of the $A^2$ term is crucial!
Within the Hopfield-Bogoliubov (or Holstein-Primakoff) approaches, the frequency spectrum of the normal phase is obtained by diagonalizing:

$$
\mathcal{M} = \begin{pmatrix}
\omega_{\text{cav}} + 2D & -i\Omega_0 & -2D & -i\Omega_0 \\
-\Omega_0 & \omega_{\text{eg}} & -i\Omega_0 & 0 \\
-2D & -i\Omega_0 & -\omega_{\text{cav}} + 2D & -i\Omega_0 \\
-i\Omega_0 & 0 & i\Omega_0 & -\omega_{\text{eg}}
\end{pmatrix}.
$$

$$
\text{Det}(\mathcal{M}) = \omega_{\text{eg}}\omega_{\text{cav}}(\omega_{\text{eg}}(4D + \omega_{\text{cav}}) - 4\Omega_0^2).
$$

$$
\text{Det}(\mathcal{M}) = 0 \implies \text{Gapless excitation = Quantum Critical Point}
$$
Cavity QED: constraints by oscillator strength sum rule

Thomas-Reiche-Kuhn oscillator strength sum rule

\[ D \geq \frac{\Omega_0^2}{\omega_{eg}} \]

Note: for electric dipole transitions
Diagrams for cavity QED (electric dipole coupling)

Dicke Model
D=0

Hopfield Model
D = $\Omega_0^2/\omega_{eg}$

No critical point

No critical point

Shifted critical point
$N$ Cooper pair boxes \textit{capacitively} coupled to a resonator

\[
H = \hbar \omega_{\text{res}} a^\dagger a + \sum_{j=1}^N \left\{ 4E_c \sum_{n \in \mathbb{Z}} (n - (\hat{n}_{\text{ext}})_j)^2 |n\rangle \langle n| - \frac{E_j}{2} \sum_{n \in \mathbb{Z}} (|n+1\rangle \langle n| + |n\rangle \langle n+1|)_j \right\}
\]

\[
(\hat{n}_{\text{ext}})_j = \frac{C_g}{2e} (V_g + \hat{V})_j
\]

Charge coupled to Cooper pair box

\[
\hat{V} = \nu(a + a^\dagger)
\]

Resonator quantum voltage (single-mode)
Interaction terms and analogy with electric dipole in cavity QED

\[ \omega_j \quad \longleftrightarrow \quad \frac{C_g}{2e} \mathcal{V} (a + a^\dagger) \sum_{j=1}^{N} (|\bar{e}\rangle \langle \bar{g}|)_j \quad + \quad \text{h.c.} = -i\hbar \tilde{\Omega}_0 (a + a^\dagger) (b + b^\dagger) \]

\[ H_{\text{coup}} = -i4E_c \frac{C_g}{2e} \mathcal{V} (a + a^\dagger) \sum_{j=1}^{N} (|\bar{e}\rangle \langle \bar{g}|)_j \quad + \quad \text{h.c.} = -i\hbar \tilde{\Omega}_0 (a + a^\dagger) (b + b^\dagger) \]

\[ H_{V^2} = \sum_{j=1}^{N} 4E_c \left( \frac{C_g}{2e} \right)^2 \mathcal{V}^2 (a + a^\dagger)^2 = \hbar \tilde{D} (a + a^\dagger)^2 \]

Analogous to $A^2$-term
Relation between $V^2$ term and vacuum Rabi frequency

For the considered system:

$$\tilde{D} = \frac{\tilde{\Omega}_0^2 E_J}{\omega_J 4E_C}.$$ 

Strong charging energy limit:

$$\frac{E_J}{4E_C} \ll 1 \Rightarrow \tilde{D} \ll \frac{\Omega_0^2}{\omega_J}.$$ 

Analogous of TRK sum rule is violated due to compact 1D wavefunction topology!

Quantum critical coupling:

$$\tilde{\Omega}^c = \frac{\sqrt{\omega_{res}\omega_J}}{2\sqrt{1 - \frac{E_J}{4E_C}}}.$$ 

Diagrams for Cooper pair boxes capacitively to resonator

Note: for other Josephson atoms capacitively coupled to a resonator the quantum phase transition can disappear ….
Conclusions

- Ultrastrong coupling regime: manipulating the QED vacuum

- Quantum phase transitions in circuit QED

- Vacuum degeneracy and finite-size scaling properties: qubits based on degenerate vacua?

- Inductive and capacitive coupling

- Circuit QED is not only analogous to cavity QED: fundamental differences can occur!