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and follow links to:

http://www.physinfo.fr/lectures.html

PDF FILES OF ALL LECTURES WILL BE POSTED ON THIS WEBSITE

Questions, comments and corrections are welcome!
CALENDAR OF SEMINARS

May 13: Denis Vion, (Quantronics group, SPEC-CEA Saclay)
Continuous dispersive quantum measurement of an electrical circuit

May 20: Bertrand Reulet (LPS Orsay)
Current fluctuations: beyond noise

June 3: Gilles Montambaux (LPS Orsay)
Quantum interferences in disordered systems

June 10: Patrice Roche (SPEC-CEA Saclay)
Determination of the coherence length in the Integer Quantum Hall regime

June 17: Olivier Buisson, (CRTBT-Grenoble)
A quantum circuit with several energy levels

June 24: Jérôme Lesueur (ESPCI)
High Tc Josephson Nanojunctions: Physics and Applications

PROGRAM OF THIS YEAR'S LECTURES

Lecture I: Introduction and overview

Lecture II: Modes of a circuit and propagation of signals

Lecture III: The "atoms" of signal

Lecture IV: Quantum fluctuations in transmission lines

Lecture V: Introduction to non-linear active circuits

Lecture VI: Amplifying quantum signals with dispersive circuits
LECTURE V: INTRODUCTION TO NON-LINEAR ACTIVE CIRCUITS

OUTLINE

1. Where we stand so far, purpose of this lecture
2. Fluctuations of damped harmonic oscillator
3. Hamiltonian approach of Caldeira and Leggett
4. Properties of scattering matrix
5. Non-linear active circuits

1ST QUANTIZATION OF SIGNALS (CLASSICAL WAVES)

Signals on a line can be represented by sum of right & left moving waves

\[ A^+ (x-v_t t) \rightarrow -A^- (x+v_t t) \]

Energy flux @ \( x = 0 \): \( \mathcal{P}(t) = \left| A^+ (t) \right|^2 - \left| A^- (t) \right|^2 \)

Decomposition of signal into modes:

\[ A_{np} \equiv \int_{-\infty}^{+\infty} dt \psi_{np}^* (t) A^- (t) \]
\[ A_{np} = (A_{np})^* \]
\[ [A_{np}] = \text{[energy]}^{1/2} \]

Each mode is a "flying oscillator":

\[ \left\{ A_{m_1 p_1}, A_{m_2 p_2} \right\}_{PB} = \frac{i}{2} \omega_{m_1} \delta_{m_1 + m_2} \delta_{p_1 p_2} \]

INFORMATION OF OSCILLATORS IS CONSERVED

\[ \int_{-\infty}^{+\infty} dt \psi_{m_1 p_1}^* (t) \psi_{m_2 p_2} (t) = \delta_{m_1 m_2} \delta_{p_1 p_2} \]
1ST QUANTIZATION OF SIGNALS: 
FRESNEL REPRESENTATION OF MODE

mode index \( \mu = \{(|p|, p) \rightarrow \text{or} \leftarrow \} \) refers to a pair of tiles

\[
A^\perp_\mu = A\{|p|p\} 
\]

In-phase and quadrature components are conjugate variables in classical sense

\[
A_{\mu}^{\perp} = A_{\mu}^{-\mu} 
\]

2ND QUANTIZATION OF SIGNALS (QUANTUM WAVES)

classical mode amplitude \( A_{mp} \rightarrow \hat{A}_{mp} \) quantum operator

Poisson bracket \( \{ A_{m_1p_1}, A_{m_2p_2} \} \rightarrow \left[ \hat{A}_{m_1p_1}, \hat{A}_{m_2p_2} \right] = \frac{\hbar \omega}{2} \delta_{m_1m_2} \delta_{p_1p_2} \) commutator

physical observables
continuous field variables

continuous field ladder operators:
(N.B.: arrows have been dropped)

\[
\hat{a}[\omega\rightarrow\omega_m] = \lim_{h\rightarrow 0} \frac{1}{\sqrt{\hbar^2 \omega_m^2}} \hat{A}_{mp}
\]

commutator of \( a \)'s:

\[
[\hat{a}[^{\omega_m}], \hat{a}[^{\omega_m}]] = \text{sgn}(\omega_1 - \omega_2) \delta[^{\omega_1 + \omega_2}]
\]

average value of anticommutator in thermal state:

\[
\langle \{ \hat{a}[^{\omega_1}], \hat{a}[^{\omega_2}] \} \rangle_T = \coth \frac{h\omega}{2k_B T} \delta[^{\omega_1 + \omega_2}]
\]
QUANTUM FLUCTUATION-DISSIPATION THEOREM

\[ \langle \hat{\mathcal{V}}^+ [\omega_1] \hat{\mathcal{V}}^+ [\omega_2] \rangle = S_{VV} [\omega_1] \delta (\omega_1 + \omega_2) \]

\[ S_{VV} [\omega] = \frac{Z}{4} \frac{\hbar \omega}{\coth (\frac{\hbar \omega}{2k_B T}) + 1} \]

\[ S_{VV} [\omega] = R \hbar \omega \left[ \coth \left( \frac{\hbar \omega}{2k_B T} \right) + 1 \right] \]

\[ S_{VV} [\omega] = 4S_{VV}^\omega [\omega] \]

\[ R = Z_c \]

QUANTUM ASYMMETRY OF FLUCTUATIONS

\( \omega > 0 \): emission into the line
\( \omega < 0 \): absorption from the line

classical value
**QUANTUM FLUCTUATION-DISSIPATION THEOREM**

\[
\langle \hat{\nu}^{\omega_1}\hat{\nu}^{\omega_2}\rangle = S_{\nu\nu}^{\omega_1\omega_2}\delta(\omega_1 + \omega_2)
\]

\[
S_{\nu\nu}^{\omega} = \frac{Z}{4} h \omega \left[ \coth \left( \frac{h \omega}{2k_B T} \right) + 1 \right]
\]

\[
S_{vy}^{\omega} = 4S_{v\nu}^{\omega} \left( \frac{2k_B T \omega}{h} \right)
\]

**Quantum Asymmetry of Fluctuations**

- \( \omega > 0 \): emission into the line
- \( \omega < 0 \): absorption from the line

**Recover results of Johnson-Nyquist noise:**

Voltage fluctuations in classical regime:

\[
S_{vy}[\nu] = 4k_BT \nu
\]

Line power flow in classical regime:

\[
P = k_BT \nu
\]

**Can we build a Quantum Spectrum Analyzer?**
CLASSICAL SPECTRUM ANALYZER

\[
A(t) \xrightarrow{\text{bandpass filter}} B[\text{Hz}] \quad \xrightarrow{\text{integrating lowpass filter}} \quad \langle |A_{\text{mp}}|^2 + |A_{\text{mp}}|^2 \rangle \\
S_{AA}(\nu) = \lim_{B \to \infty} \left( \langle |A_{\text{mp}}|^2 + |A_{\text{mp}}|^2 \rangle \right)
\]

Wiener-Kinchin Theorem:
\[
S_{AA}(\nu) = 2S_{AA}[\omega = 2\pi\nu]
\]

QUANTUM MECHANICALLY \(A(t_1)\) AND \(A(t_2)\) DO NOT COMMUTE!

QUANTUM SPECTRUM ANALYZER

spin 1/2 in magnetic field:
\[
\hat{S} = \delta \bar{B}(t) / \hat{\lambda}
\]

F.G.R.:
\[
\Gamma_{1=0} = \frac{2\pi}{\hbar} |\lambda \langle 0 | \sigma_3 | 1 \rangle|^2 \quad S_{AA}[\omega = 0]
\]

ABSORPTION
\[
\omega_{\text{in}} = -\omega_B
\]

EMISSION
\[
\omega_{\text{out}} = \omega_B
\]
PURPOSE OF THIS LECTURE

WHAT IS THE EFFECT OF THE QUANTUM PART OF FLUCTUATIONS?

DIFFERENT CLASSES OF RESPONSE OF CIRCUITS

- DISPERSIVE vs DISSIPATIVE
- PASSIVE vs ACTIVE
- LINEAR vs NON-LINEAR

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SCATTERING APPROACH TO DRIVEN DISSIPATIVE CIRCUITS

MAIN IDEA: RESISTANCE IS EQUIVALENT TO SEMI-INFINITE TRANSMISSION LINE

\[ V_0 \cos(\omega_0 t) \]

\[ L \quad C \quad R \]

\[ \cos(\omega_0 t) \]

\[ I_0 \cos(\omega_0 t) \]

\[ L \quad C \quad R \]

\[ Z_c = R \]

\[ A^+ (t) : A^+ (t) \]

\[ A^{in} (t) : A^- (t) \]

\[ A^{in} (t) \]

\[ A^- (t) \]

\[ A^{out} (t) \]

\[ \text{DRIVE CAN BE TREATED AS INCOMING SIGNAL} \]

UNLIKE \( A^+ (t) \) AND \( A^- (t) \), \( A^{in} (t) \) AND \( A^{out} (t) \) ARE NOT INDEPENDENT

TERMINATING CIRCUIT IMPOSES A STRICT RELATIONSHIP BETWEEN INCOMING AND OUTGOING WAVES

A KEY IDEA

\[ A^+ (t) \]

\[ Z_c = R \]

\[ A^+ (t) \]

\[ A^- (t) \]

\[ A^{in} (t) \]

\[ A^{out} (t) \]
SAME CIRCUIT, DIFFERENT POINTS OF VIEW

DAMPED, DRIVEN STANDING OSCILLATOR

\[ V(t) \rightarrow I_D(t) \rightarrow V(t) \]

IMPEDANCE

PURELY DISPERSIVE SCATTERER OF WAVES

\[ A^\text{in}(t) \rightarrow A^\text{out}(t) \]

SCATTERING MATRIX

RESISTANCE-TRANSMISSION LINE EQUIVALENCE

\[
\begin{align*}
V(t) &= \Phi(t) = R(I_D(t) + I_N(t) - I(t)) \\
C\dot{\Phi} + \frac{\dot{\Phi}}{R} + \frac{\Phi}{L} &= I_D(t) + I_N(t) \\
\ddot{\Phi} + \frac{\dot{\Phi}}{RC} + \frac{\Phi}{LC} &= \frac{I_D(t) + I_N(t)}{C} \\
V(t) + Z_I(t) &= 2V^\text{in}(t) \\
\ddot{\Phi} + 2\Gamma\dot{\Phi} + \omega_0^2\Phi &= 4\Gamma V^\text{in}(t) \\
V^\text{out}(t) &= \Phi - V^\text{in}(t)
\end{align*}
\]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \text{ resonance frequency} \]
\[ \Gamma = \frac{1}{2RC} \text{ amplitude damping rate (HWHM)} \]
\[ \frac{\omega_0}{2\Gamma} = Q \gg 1 \text{ quality factor} \]
**INPUT-OUTPUT RELATION IN OPERATOR FORM**

\[ \ddot{\Phi} + 2\Gamma \dot{\Phi} + \omega_0^2 \Phi = 4\Gamma \dot{V}^{in}(t) \]

\[ V^{\text{out}}(t) = \Phi - \dot{V}^{in}(t) \]

\[ \frac{d^2}{dt^2} \Phi + 2\Gamma \frac{d}{dt} \Phi + \omega_0^2 \Phi = 4\Gamma \dot{\hat{V}}^{in}(t) \]

\[ \dot{\hat{V}}^{\text{out}}(t) = \frac{d}{dt} \dot{\hat{\Phi}} - \dot{\hat{V}}^{in}(t) \]

Fourier domain:

\[ (\omega_0^2 - \omega^2 + 2i\omega \Gamma) \hat{\Phi}[\omega] = 4\Gamma \dot{\hat{V}}^{in}[\omega] \]

\[ \dot{V}^{\text{out}}[\omega] = -\frac{\omega_0^2 - \omega^2 - 2i\omega \Gamma}{\omega_0^2 - \omega^2 + 2i\omega \Gamma} \hat{\Phi}[\omega] - \hat{V}^{in}[\omega] \]

**UNIT MODULUS COMPLEX NUMBER**

**ROTATING WAVE APPROXIMATION**

Since \( \frac{\Gamma}{\omega_0} = \kappa = \frac{1}{2Q} \ll 1 \) we can consider 1 pole at a time

\[ r(\omega) = -\frac{\omega_0^2 - \omega^2 - 2i\omega \Gamma}{\omega_0^2 - \omega^2 + 2i\omega \Gamma} \approx \frac{1 - i\frac{\omega^2 - \omega_0^2}{\Gamma}}{1 + i\frac{\omega^2 - \omega_0^2}{\Gamma}} \]

\[ V[\omega] = (1 + r[\omega])V^{in}[\omega] \approx \frac{2}{1 + i\frac{\omega - \omega_0}{\omega_0}} V^{in}[\omega] \]

**example:** resonant drive

\[ \omega = \omega_0 \]

\[ \omega - \omega_0 \]

\[ \omega_0 \]

\[ r(\omega) \]

\[ \frac{\omega_0}{2\pi} \]
FLUCTUATIONS OF THE RESISTIVELY DAMPED LC RESONATOR

\[
\langle \Phi[-\omega] \Phi[\omega] \rangle = \frac{1 + r(\omega)}{\omega^2} S_{\eta \eta}^{in}[\omega]
\]
\[
= Z_c \frac{1 + r(\omega)}{\omega^2} \left( \frac{\hbar \omega}{2} \coth \frac{\hbar \omega}{2k_bT} \right)
\]

\[
\langle \Phi^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \langle \Phi[-\omega] \Phi[\omega] \rangle
\]
converges

\[
\langle \hat{Q}^2 \rangle = \frac{1}{2\pi Z_0^2} \int_{-\infty}^{+\infty} d\omega \omega^2 \langle \Phi[-\omega] \Phi[\omega] \rangle
\]
diverges!

\[
Z_a = \frac{1}{\sqrt{C}}
\]

INTEGRAL CAN BE PERFORMED ANALYTICALLY AND YIELD MEANINGFUL RESULTS


\[
\langle \Phi^2 \rangle = \frac{k_bT}{2L}
\]

\[
\langle \Phi^2 \rangle = \frac{\hbar \omega_0}{hZ_0}
\]

\[
\Psi(x) \text{ is polygamma function}
\]

\[
\varphi_\kappa = \frac{\kappa \pm \sqrt{\kappa^2 - 1}}{2\pi k_b T / (\hbar \omega_0)}
\]

\[
\kappa = 0.02 = 0.2 = 2 = 20
\]

\[
\frac{k_bT}{\hbar \omega_0} = 0.02 \quad 0.2 \quad 2 \quad 20
\]

suppression of flux fluctuations by shunt
CHARGE FLUCTUATIONS OF THE DAMPED LC RESONATOR WITH INDUCTIVE CUTOFF

\[ \omega_c = \frac{R}{L_c} \]

\[ \langle \dot{Q}^2 \rangle = \frac{1}{Z_0} \langle \Phi^2 \rangle + \Delta_c \]

\[ \Delta_c = \frac{\hbar c}{\pi Z_0} \left[ 2 \Psi (1 + \lambda_c) - \frac{1}{\sqrt{\lambda_c^2 - 1}} \left( \dot{\lambda}_c \Psi (1 + \lambda_c) - \dot{\lambda}_c \Psi (1 + \lambda_c) \right) \right] \]

\[ \dot{\lambda}_c = \frac{\hbar \omega_0}{2\pi k_B T} \left( \frac{\omega}{\omega_0} - 2\kappa \right) \]

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ANOTHER POINT OF VIEW: HAMILTONIAN APPROACH OF CALDEIRA AND LEGGETT

\[
\begin{align*}
Z[\omega] & = \frac{L_\mu}{\sqrt{L_\mu C_\mu}} \\
\text{Re}[Z[\omega]] & = \frac{1}{\sqrt{L_\mu C_\mu}}
\end{align*}
\]

standing oscillator modes; nb. of oscillators \( N \rightarrow \infty \)

\[
\begin{align*}
Y[\omega] & = \frac{C_\mu}{L_\mu} \\
\text{Re}[Y[\omega]] & = \frac{1}{\sqrt{L_\mu C_\mu}}
\end{align*}
\]

GENERALIZATION OF QUANTUM FLUCTUATION-DISSIPATION THEOREM TO ARBITRARY IMPEDANCE OR ADMITTANCE

\[
\begin{align*}
S_{VV}[\omega] & = \hbar \omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) + 1 \text{ Re}(Z[\omega]) \\
S_{II}[\omega] & = \hbar \omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) + 1 \text{ Re}(Y[\omega])
\end{align*}
\]

QUANTUM BLACK-BODY BOSE FACTOR

CLASSICAL RESPONSE FUNCTION
GENERALIZATION OF QUANTUM FLUCTUATION-DISSIPATION THEOREM TO ARBITRARY IMPEDANCE OR ADMITTANCE

\[ S_{VV} [\omega] = \hbar \omega \left[ \coth \left( \frac{\hbar \omega}{2k_B T} \right) + 1 \right] \Re \left( Z [\omega] \right) \]

\[ S_{II} [\omega] = \hbar \omega \left[ \coth \left( \frac{\hbar \omega}{2k_B T} \right) + 1 \right] \Re \left( Y [\omega] \right) \]

FLUCTUATIONS OF OSCILLATOR FOR AN ARBITRARY ADMITTANCE

\[ \hat{H} = \frac{\Phi^2}{2L} + \frac{\hat{Q}^2}{2C} + \sum_{\mu=1}^{N} \left[ \frac{\hat{\Phi}_{\mu}^2 - \Phi^2}{2C_\mu} \right] \]

\[ Z_{\omega} [\omega] = \frac{1}{j\omega + jC\omega + Y[\omega]} \quad (j = -i) \]

\[ \left\langle \Phi^2 \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{\hbar Z_\omega [\omega]}{2\omega} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \]

\[ \left\langle \hat{Q}^2 \right\rangle = \frac{C^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{\hbar \omega Z_\omega [\omega]}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \]

08-V-22a

08-V-22b
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MULTIPLE PORT CIRCUITS

Example:

\[ A_{in}^{out}(t) \rightarrow A_{out}^{in}(t) \]

In general:

\[ \text{port 1} \rightarrow \text{port 2} \rightarrow \text{port 3} \]
SCATTERING MATRIX FOR PASSIVE LINEAR CIRCUIT

\[
\begin{bmatrix}
A_1^{\text{out}}(\omega) \\
A_2^{\text{out}}(\omega) \\
\vdots \\
A_t^{\text{out}}(\omega)
\end{bmatrix}
= \begin{bmatrix}
s_{11}(\omega) & s_{12}(\omega) & \cdots & s_{1t}(\omega) \\
s_{21}(\omega) & s_{22}(\omega) & \cdots & s_{2t}(\omega) \\
\vdots & \vdots & \ddots & \vdots \\
s_{t1}(\omega) & s_{t2}(\omega) & \cdots & s_{tt}(\omega)
\end{bmatrix}
\begin{bmatrix}
A_1^{\text{in}}(\omega) \\
A_2^{\text{in}}(\omega) \\
\vdots \\
A_t^{\text{in}}(\omega)
\end{bmatrix}
\]

OUTGOING AMPLITUDES \rightarrow “S” PARAMETERS, SCATTERING MATRIX \rightarrow INCOMING AMPLITUDES

CAUSALITY \rightarrow POLES OF S MATRIX IN LOWER-HALF PLANE

INFORMATION CONSERVATION + ENERGY CONSERVATION \rightarrow UNITARITY OF S MATRIX \quad S^†S = 1

SCATTERING MATRIX FOR NON-DISSIPATIVE LINEAR 2-PORT

\[
S = \begin{bmatrix}
e^{i\alpha} & e^{i\beta}\sqrt{1-r^2} \\
e^{i\gamma} & e^{i\delta}\sqrt{1-r^2}
\end{bmatrix}
\]

Unitarity constraint: \quad \alpha + \delta - \beta - \gamma = \pi \text{ mod } 2\pi

Fully balanced 2-port:

Note that \(S_{11}\) and \(S_{22}\) are reflection coefficients for port 1 and 2 provided the other port is terminated by characteristic impedance.
SYMMETRIES OF S MATRIX: CONSTRAINTS FOR LINEAR 3-PORT (NO PERFECT DIVIDER EXISTS)

THEOREM: IT IS IMPOSSIBLE FOR A NON-DISSIPATIVE 3-PORT TO BE SIMULTANEOUSLY POWER-MATCHING AND RECIPROCAL.

POWER-MATCHING: \( \forall l, s_{ll} = 0 \)
RECIPROCITY: \(^t S = S\)


- Non-dissipative and reciprocal but not power matching:
  \[ s_{22} = s_{33} \neq 0 \]

- Non-dissipative and power-matching but not reciprocal:
  Circulator:

- Reciprocal and power-matching but dissipative:
  Wilkinson divider:

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**ACTIVE LINEAR 1-PORT**

Simplest example:

\[ L(t) = L + \delta L \sin(2\omega_b t) \]

*Degenerate Parametric Amplifier*

\[ \ddot{\Phi} + 2\Gamma \dot{\Phi} + \omega_0^2 \Phi + i\omega_0^2 \chi \Phi \left( e^{i\omega_0 t} - e^{-i\omega_0 t} \right) = 4\Gamma V_{\text{in}}(t) \]

\[ V_{\text{out}}(t) = \Phi - V_{\text{in}}(t) \]

Harmonic balance

\[
2i\omega_b \Gamma \Phi \left[ \omega_0 \right] - i\omega_0^2 \chi \Phi \left[ -\omega_0 \right] = 4\Gamma V_{\text{in}}(t) \left[ \omega_0 \right]
\]

After a few steps:

\[
\begin{align*}
A^{\omega} \left[ \omega_0 \right] &= \left( 1 + \frac{c^2}{1 - s^2} \right) A^{\omega} \left[ \omega_0 \right] + \frac{2s}{1 - s^2} A^{\omega} \left[ -\omega_0 \right] \\
A^{\omega} \left[ -\omega_0 \right] &= \left( 1 + \frac{s^2}{1 - s^2} \right) A^{\omega} \left[ -\omega_0 \right] + \frac{2s}{1 - s^2} A^{\omega} \left[ +\omega_0 \right]
\end{align*}
\]

Generalized scattering matrix

\[
\begin{bmatrix}
A^{\omega} \left[ \omega_0 \right] \\
A^{\omega} \left[ -\omega_0 \right]
\end{bmatrix} =
\begin{bmatrix}
c & s \\
s & c
\end{bmatrix}
\begin{bmatrix}
A^{\omega} \left[ \omega_0 \right] \\
A^{\omega} \left[ -\omega_0 \right]
\end{bmatrix}
\]

\[ c^2 - s^2 = 1 \]

---

**NON-LINEAR CIRCUIT CAN BEHAVE LINEARLY FOR WEAK SIGNALS: SQUID**

\[ L(t) = L + \delta L \sin(2\omega_b t) \]

\[ I_B(t) = I_B + I_F \sin(2\omega_D t) \]

\[ L_S^{\text{SQUID}} \]

IN THE VICINITY OF OPERATING POINT

SQUID IS A VARIABLE INDUCTOR

### SCATTERING MATRIX FOR ACTIVE LINEAR 2-PORTS

\[
\begin{bmatrix}
    a_1^{\text{out}}(\omega_1) \\
    a_1^{\text{out}}(-\omega_1) \\
    a_2^{\text{out}}(\omega_2) \\
    a_2^{\text{out}}(-\omega_2)
\end{bmatrix}
= \begin{bmatrix}
    r_1 & s_1 & t_{12} & u_{12} \\
    s_1^* & r_1^* & u_{12}^* & t_{12}^* \\
    t_{21} & u_{21} & r_2 & s_2 \\
    u_{21}^* & t_{21}^* & s_2^* & r_2^*
\end{bmatrix}
\begin{bmatrix}
    a_1^{\text{in}}(\omega_1) \\
    a_1^{\text{in}}(-\omega_1) \\
    a_2^{\text{in}}(\omega_2) \\
    a_2^{\text{in}}(-\omega_2)
\end{bmatrix}
\]

\[a(\omega) \sim \frac{1}{\sqrt{\omega}} A(\omega)\]

Information conservation → S matrix is \(\mathbf{SJS}=\mathbf{J}\)

(energy is not conserved, \(\mathbf{S}\) is not unitary)

\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

Central to antisymmetric bilinear forms such as Poisson Brackets (and commutators)

### IN NON-LINEAR ACTIVE DEVICES, SIGNALS CAN SCATTER BETWEEN DIFFERENT FREQUENCIES

\[
\begin{array}{c}
-\omega_2 \\
-\omega_1 \\
0 \\
\omega_1 \\
\omega_2 \\
-\omega_1 \\
-\omega_2 \\
0 \\
\end{array}
\]

\[
\begin{array}{c}
\text{IN} \\
\omega \\
\text{OUT}
\end{array}
\]

Pump frequency: \(\Omega_p\)

\[
\omega_1 + \omega_2 = 0 \mod \Omega_p
\]
TOPICS OF NEXT LECTURE:

1) AMPLIFICATION OF QUANTUM SIGNALS: ULTIMATE SENSITIVITY

2) SQUEEZING OF QUANTUM NOISE

END OF LECTURE