INTRODUCTION AU CALCUL QUANTIQUE

INTRODUCTION TO QUANTUM COMPUTATION

Quatrième Leçon / Fourth Lecture

VISIT THE WEBSITE OF THE CHAIR OF MESOSCOPIC PHYSICS

http://www.college-de-france.fr
then follow
Enseignement > Sciences Physiques > Physique Mésoscopique > Site web
or
http://www.physinfo.fr/lectures.html

PDF FILES OF ALL PAST LECTURES ARE POSTED

Questions, comments and corrections are welcome!
write to "phymeso@gmail.com"
CONTENT OF THIS YEAR'S LECTURES

QUANTUM COMPUTATION FROM THE PERSPECTIVE OF MESOSCOPIC CIRCUITS

1. Introduction, c-bits versus q-bits
2. The Pauli matrices and quantum computation primitives
3. Stabilizer formalism for state representation
4. Clifford calculus
5. Algorithms
6. Error correction

CALENDAR OF SEMINARS

**May 11:** Cristian Urbina, (Quantronics group, SPEC-CEA Saclay)
Josephson effect in atomic contacts and carbon nanotubes

**May 18:** Benoît Douçot (LPTHE / Université Pierre et Marie Curie)
Towards the physical realization of topologically protected qubits

**June 1:** Takis Kontos (LPA / Ecole Normale Supérieure)
Points quantiques et ferromagnétisme

**June 8:** Cristiano Ciuti (MPQ, Université Paris - Diderot)
Ultrastrong coupling circuit QED: vacuum degeneracy and quantum phase transitions

**June 15:** Leo DiCarlo (Yale)
Preparation and measurement of tri-partite entanglement in a superconducting quantum circuit

**June 22:** Vladimir Manucharian (Yale)
The fluxonium circuit: an electrical dual of the Cooper-pair box?
LECTURE IV : CLIFFORD CALCULUS

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations

OUTLINE

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations
BASIC INGREDIENTS OF STABILIZER FORMALISM

Primary Pauli operators: \[ P_1, P_2, \ldots, P_N \quad P_i \in \{ I, Z, X, Y \} \]

\( N \): number of qubits

\( N = 6 \) example: \( \text{ZIZIXX} \)

Stabilizer element: \[ \pm P_1 P_2 \ldots P_N = M \neq -I \]

Stabilizer: \[ \left\{ M_1, M_2, \ldots, M_{2^N-1} \right\} \quad \forall \{ j, k \}, M_j M_k M_j M_k = I \]

represents a state of the register

Condensed form: \[ \left\{ M_\alpha, M_\beta, \ldots, M_\nu \right\} \quad \forall \{ \alpha, \beta, \gamma \}, M_\alpha M_\beta \neq \pm M_\gamma \]

\( N \): number of qubits

\( P \): number of elements

A D-dimensional state manifold of the register is represented by:

\[ \left\{ M_\alpha, M_\beta, \ldots, M_\mu \right\} \quad D = N - P \]

STABILIZER ↔ REGISTER STATE

A single-state stabilizer can be written as an array of symbols:

\[ \begin{pmatrix}
\pm P_{11} & P_{12} & \ldots & P_{1N} \\
\pm P_{21} & P_{22} & \ldots & P_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\pm P_{N1} & P_{N2} & \ldots & P_{NN}
\end{pmatrix} \]

N Pauli words with N+1 symbols arranged in "alphabetical" order (I-Z-X-Y)

The associated state simultaneously diagonalizes the operator-words with +1 eigenvalue:

\[ +IZ \quad |00\rangle \quad +IX \quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |++\rangle \quad -ZZ \quad |01\rangle - |10\rangle \]

\[ +IZ \quad |10\rangle \quad -IX \quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |+\rangle \quad -ZZ \quad |01\rangle + |10\rangle \]

10-IV-7c
**STABILIZER – STATE MAPPING (continued)**

\[
+ZZ + XX \quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)}{2\sqrt{2}} = \frac{+ + - -}{\sqrt{2}}
\]

remove signs \[ ZZ \quad XX \]

\[
\rightarrow \quad \{ \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}} \}
\]

less is more!

\[
+ZX - XZ \quad \rightarrow \quad ? \quad \text{Observe that:}
\]

\[
+ZZ + XX \quad R_{y,\pi/2} \rightarrow +ZX - XZ
\]

since \[ \pi/2 \text{ rotation around } Y \]

\[
|00\rangle + |01\rangle - |10\rangle + |11\rangle
\]

**STABILIZER – STATE MAPPING (continued)**

\[
+IZ + ZI + ZZ \quad |00\rangle \quad +ZZ + XX \quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad +ZZ - XX \quad \frac{|00\rangle - |11\rangle}{\sqrt{2}}
\]

pivot \[ XX|00\rangle = |11\rangle \]
\[ XX|11\rangle = |00\rangle \]

\[
+IIZ + IIZ + ZII \quad |000\rangle \quad +IIZ + ZZ + XXX \quad \frac{|000\rangle + |111\rangle}{\sqrt{2}} \quad +IIZ + ZZ - XXX \quad \frac{|000\rangle - |111\rangle}{\sqrt{2}}
\]

pivot \[ XXX|000\rangle = |111\rangle \]
\[ XXX|111\rangle = |000\rangle \]

10-IV-9
OUTLINE

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations

CONSTRUCTION OF A STABILIZER CLASS

All the Pauli multi-qubit primary operators

$I$, $IIZ$, $IIX$, $IIY$, $YYI$
CONSTRUCTION OF A STABILIZER CLASS

pick one operator: $IIZ$

exclude non-com. operators
CONSTRUCTION OF A STABILIZER CLASS

- I
- X
- Y

1. Pick next operator.
2. Exclude non-commuting operators.

10-IV-10d
NUMBER OF STABILIZER CLASSES

First choice: $2^{2N} - 1$  
the -1 comes from excluding $I$

Second choice: $2^{2N-1} - 2$  
exclude non.com., last gen. and $I$

Third choice: $2^{2N-2} - 4$  
exclude non.com., last gen. comb. and $I$

$k$-th choice: $2^{2N-k+1} - 2^{k-1}$

Total nb. choices: $\prod_{k=1}^{N} (2^{N-k+1} - 2^{k-1}) = \prod_{k=1}^{N} (2^{N-k+1} + 1) \left( 2^N - 2^{k-1} \right)$

Last factor corresponds to the number of arrangements of numbers 1 thru N on hyper-cube of stabilizer gen. combinations. It must be divided out to get the classes.

The number of stabilizer classes is thus:

$$\prod_{k=1}^{N} (2^{N-k+1} + 1) = \left( 2^N + 1 \right) \left( 2^{N-1} + 1 \right) \cdots 5 \cdot 3$$

NUMBER OF CLIFFORD STATES

$N_{CS} = 2^N \left( 2^N + 1 \right) \left( 2^{N-1} + 1 \right) \cdots 5 \cdot 3$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N_{CS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>1080</td>
</tr>
<tr>
<td>4</td>
<td>36720</td>
</tr>
<tr>
<td>5</td>
<td>2423520</td>
</tr>
</tbody>
</table>

Information in selecting one of these states: $\ln_2 N_{CS}$

$\ln_2 N_{CS}$ grows like $N^2/2$

Information is super-extensive!

Gottesman-Knill theorem

entanglement, Bell violations, teleportation, error correction, but...
OUTLINE

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations

THE SINGLE QUBIT CLIFFORD GROUP

Consider isomorphisms of the Pauli group into itself

Each isomorphism is characterized by the set of the images of the generators: \( \{ g_z, g_x \} \in \{ Z, -Z, X, -X, Y, -Y \}^{\times 2} \)

\[
\begin{align*}
Z &\mapsto g_z \\
X &\mapsto g_x \\
g_z g_x &= -g_x g_z
\end{align*}
\]

The two images must satisfy:

There are therefore 6x4=24 isomorphisms, called elements of the 1-qubit Clifford group:

\[
Id = \left[ \begin{array}{c} Z \\ X \end{array} \right] \quad [Z]^{1/2} = \left[ \begin{array}{c} Z \\ X \end{array} \right] \\
H = \left[ \begin{array}{c} Z \\ X \end{array} \right] \quad [Z]^{-1/2} = \left[ \begin{array}{c} Z \\ X \end{array} \right] \\

Why \([P]^{1/2}\) and not \([P]^{1/2}\)? The latter is an operator acting on kets. The former is a super-operator acting on operators.
THE SINGLE QUBIT CLIFFORD GROUP IS ISOMORPHIC TO THE OCTAHEDRAL GROUP

The notation \([A]^{\alpha}\) means the transformation is a "rotation" around axis \(A\) with angle \(\alpha \pi\). Attention: \([A]^{\alpha} = Id\)

GENERATORS OF THE 1-QUBIT CLIFFORD GROUP (ISOMORPHIC TO THE OCTAHEDRAL GROUP)

Consider \(S_4\) the permutation group on 4 objects, isomorphic to the octahedral group, symmetry group of the cube and the octahedron

1st choice of generators
\[
R_x(\pi/2) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix} = [Z]^{1/2}
\]
\[
R_{x+z}(\pi) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{bmatrix} = [H]
\]

2nd choice of generators
\[
R_x(\pi/2) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix} = [Z]^{1/2}
\]
\[
R_{x+z}(\pi) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = [X]^{1/2}
\]
USEFUL RELATIONS FOR QUANTUM COMPILERS

\[
\begin{align*}
\langle Y \rangle^{1/2} &= \langle Z \rangle^{-1/2} \langle X \rangle^{1/2} \langle Z \rangle^{1/2} \\
\langle H \rangle &= \langle Z \rangle^{1/2} \langle X \rangle^{1/2} \langle Z \rangle^{1/2}
\end{align*}
\]

OUTLINE

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations
UPGRADING THE PROCESSOR TO 4 QUBITS

CIRCUIT HAMILTONIAN

\[ H_1 = \omega_1 a_1^+ a_1 + \frac{\alpha_1}{2} (a_1^+ a_1)^2 \]

\[ H_2 = \omega_2 a_2^+ a_2 + \frac{\alpha_2}{2} (a_2^+ a_2)^2 \]
\[ H_{\text{total}} = H_1 + H_2 + H_c + H_{\text{coupling}} \]

\[ H_1 = \frac{\alpha_1 a_1^\dagger a_1 + \alpha_1}{2} (a_1^\dagger a_1)^2 \]

\[ H_2 = \frac{\alpha_2 a_2^\dagger a_2 + \alpha_2}{2} (a_2^\dagger a_2)^2 \]

\[ H_c = \omega_c a_c^\dagger a_c \]

\[ H_{\text{coupling}} \simeq g_1 (a_1^\dagger a_2^\dagger + a_2 a_1^\dagger) + g_2 (a_2^\dagger a_2 + a_2 a_2^\dagger) \]
**EFFECTIVE QUBIT-QUBIT INTERACTIONS**

"FLIP-FLOP"

"SECULAR"

**NATURAL ENTANGLING OPERATIONS**

\[ \vec{\sigma}_1 \quad \vec{\sigma}_2 \]

\[ \hat{U}(\tau) = \exp \left( -i \hat{H}_{\text{int}} \tau / \hbar \right) \]

* Secular interaction:
  \[ \hat{H}_{\text{int}} = g_\| \sigma_z^1 \sigma_z^2 \]
  \[ \rightarrow [ZZ]^{1/2} \]
  with adjustment of gate duration time: \[ \tau_s = \frac{\pi \hbar}{4g_\|} \]

* Flip-flop interaction:
  \[ \hat{H}_{\text{int}} = g_\perp \left( \sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2 \right) \]
  \[ = g_\perp \left( \sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2 \right) / 2 \]
  \[ \tau_f = \frac{\pi \hbar}{4g_\perp} \]
  \[ \rightarrow [XX]^{1/4} [YY]^{1/4} \]
REFOCUS SEQUENCE NECESSARY WITH FLIP-FLOP INTERACTION

Consider \( \hat{H}_{\text{int}} = g_x(t) \sigma_{1x} \sigma_{2x} + g_y(t) \sigma_{1y} \sigma_{2y} \)

Green points: \([ZZ]^{1/2}\) equivalent

Need to cut sequence in 2 sections, with a 180° flip of one qubit around x or y.

OUTLINE

1. Review of stabilizer properties
2. Number of stabilizers
3. One-qubit Clifford operations
4. Two-qubit interactions
5. Two-qubit Clifford operations
### Gottesman Tables for Quantum Operations

<table>
<thead>
<tr>
<th>Bit Flip (NOT)</th>
<th>Phase Flip ( \pi ) rotation along ( \hat{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi ) rotation along ( \hat{x} )</td>
<td>( \pi ) rotation along ( \hat{x} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hadamard ( \pi ) rotation along ( \hat{x} + \hat{z} )</th>
<th>NOT(^{1/2} ) ( \pi/2 ) rotation along ( \hat{y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>( Y ) ( 1/2 )</td>
</tr>
</tbody>
</table>

#### Example of a 2-qubit gate:

\[
|c, t\rangle \rightarrow |c, t \oplus c\rangle
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>CNOT</th>
<th>phase kick-back</th>
</tr>
</thead>
<tbody>
<tr>
<td>( IZ ) ( ZZ )</td>
<td>( ZI ) ( ZI )</td>
</tr>
<tr>
<td>( IX ) ( IX )</td>
<td>( XI ) ( XX )</td>
</tr>
</tbody>
</table>

---

### Rules of Clifford Calculus for N Qubits

They are found starting from: \( [B]^\alpha A = [B]^{-\alpha} A [B]^\alpha \)

<table>
<thead>
<tr>
<th>( [B]^1 [A] = [-A] )</th>
<th>if ( A ) and ( B ) anticommute</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = [A] )</td>
<td>if ( A ) and ( B ) commute</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( [B]^{1/2} [A] = [B]^1 [A] )</th>
<th>if ( A ) and ( B ) anticommute</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = [A] )</td>
<td>if ( A ) and ( B ) commute</td>
</tr>
</tbody>
</table>

**NOTE THAT:** \( [B]^\alpha = [B]^{-\alpha} \) \( [A] = [B] \leftrightarrow A = B \)

---

10-IV-24a
EXAMPLE OF CLIFFORD CALCULATIONS

\[
\begin{align*}
[IZ]^{1/2} \, IX &= IY \\
[IZ]^{1/2} \, IY &= -IX \\
[IY]^{1/2} \, IZ &= IX \\
[IY]^{1/2} \, IX &= -IZ \\
[ZZ]^{1/2} \, IX &= ZY \\
[ZZ]^{1/2} \, XI &= YZ \\
[ZZ]^{1/2} \, XX &= XX \\
[ZZ]^{1/2} \, YY &= YY
\end{align*}
\]

\[
\begin{align*}
[IZ]^{1/2} \, IX &= IX \\
[IY]^{1/2} \, IZ &= IZ \\
[IY]^{1/2} \, XI &= XI \\
[IY]^{1/2} \, IX &= -XI \\
[ZZ]^{1/2} \, IX &= ZI \\
[ZZ]^{1/2} \, XI &= ZI \\
[ZZ]^{1/2} \, XX &= XI \\
[ZZ]^{1/2} \, YY &= XI
\end{align*}
\]

Only need to know:
\[
\hat{z} \times \hat{x} = \hat{y}
\]

EXAMPLE OF POWER OF STABILIZER FORMALISM (1)

How do we go from the Computational basis to the Sign basis?

\[
\{IZ, ZI, ZZ\} \quad \rightarrow \quad \{IX, XI, XX\}
\]

\[
\begin{align*}
\{00, 01, 10, 11\} &= \{++ , +-, -+, -\} \\
\end{align*}
\]

For both qubit, Z must be changed into X
This is performed by a 90° rotation around Y (easier than H).

\[
\begin{align*}
[IZ]^{1/2} \, IX &= IX \\
[IY]^{1/2} \, IZ &= IZ \\
[IY]^{1/2} \, XI &= XI \\
[IY]^{1/2} \, IX &= -XI
\end{align*}
\]
EXAMPLE OF POWER OF STABILIZER FORMALISM (2)

How do we go from the i-Sign basis to the Phase basis?

\[ \{ IY, YI, YY \} \quad \overset{\text{}}{\longrightarrow} \quad \{ ZX, XZ, YY \} \]

\[
\begin{align*}
\left[ |0\rangle + |1\rangle \right] |0\rangle |0\rangle, & \quad \left[ |0\rangle - |1\rangle \right] |0\rangle |0\rangle \\
\left[ |0\rangle + |1\rangle \right] |1\rangle |1\rangle, & \quad \left[ |0\rangle - |1\rangle \right] |1\rangle |1\rangle \\
\end{align*}
\]

\[ \left[ |0\rangle + |1\rangle \right] |0\rangle |1\rangle, \quad \left[ |0\rangle - |1\rangle \right] |0\rangle |1\rangle \]

\[ \left[ |0\rangle + |1\rangle \right] |1\rangle |0\rangle, \quad \left[ |0\rangle - |1\rangle \right] |1\rangle |0\rangle \]

\[ \left[ |0\rangle + |1\rangle \right] |1\rangle |1\rangle, \quad \left[ |0\rangle - |1\rangle \right] |1\rangle |1\rangle \]

We need an entanglement operator: \[ ZZ \]^\frac{1}{2}

\[ ZZ \]^\frac{1}{2} IY = -ZX \]

\[ ZZ \]^\frac{1}{2} YI = -XZ \]

\[ ZZ \]^\frac{1}{2} YY = YY \]

signs unimportant in basis representation

VISUALIZING THE CLIFFORD MOVES ON THE STABILIZER MAP (1)
VISUALIZING THE CLIFFORD MOVES ON THE STABILIZER MAP (2)

END OF LECTURE