AMPLIFICATION ET RETROACTION QUANTIQUES

QUANTUM AMPLIFICATION AND FEEDBACK

Seconde Leçon / Second Lecture

Transparents des leçons disponibles à http://www.physinfo.fr/lectures.html

PROGRAM OF THIS YEAR'S LECTURES

Lecture I: Introduction to quantum-limited amplification and feedback

Lecture II: How do we model open, out-of-equilibrium, non-linear quantum systems?

Lecture III: Is it possible to optimize the parametric amplifier characteristics while maintaining its noise at the quantum limit?

Lecture IV: What are the minimal requirements for an active circuit to be fully directional and noiseless?

Lecture V: Can continuous quantum measurements be viewed as a form of Brownian motion?

Lecture VI: How can we maintain a dynamic quantum state alive?

Please note that there will be no lecture on May 24
**LECTURE II : MODELLING OPEN, OUT-OF-EQUILIBRIUM, NON-LINEAR QUANTUM CIRCUITS**

**OUTLINE**

1. Modes of isolated, linear quantum circuits, non-linear processes
2. Open, out-of-equilibrium, linear systems: input-output theory
3. Characterizing non-linear elements, participation ratio
CLOSED, NON-DISSIPATIVE LINEAR CIRCUITS

Example:
- just inductances and capacitances
- no sources and no resistances
→ undamped normal modes

# of modes = # independent pairs of capacitances and inductances

Levels:

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Hamiltonian:

\[ H = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + \hbar \omega_c c^\dagger c \]
\[ = \sum_{m=a,b,c} \hbar \omega_m a_m^\dagger a_m \]

- \( m \) : normal mode index
- \( a_m \) : normal mode amplitude

LUMPED ELEMENTS MICROWAVE CIRCUITS

Nb on Sapphire

Capacitance

Inductance

Feedline

Nb ground plane

f = 6GHz

500μm
**HAMILTONIAN FROM CHARGES AND FLUXES**

Use loop variables:

\[
\Phi_i^{\text{loop}} = \int_{t_0}^t V_i^{\text{ind}} \, dt,
\]

\[
Q_i^{\text{loop}} = \int_{t_0}^t I_i^{\text{cap}} \, dt.
\]

Hamiltonian:

\[
H = \frac{\Phi_1^2}{2L_1} + \frac{\Phi_2^2}{2L_2} + \frac{\Phi_3^2}{2L_3} + \frac{(Q_1 + Q_2)^2}{2C_1} + \frac{(Q_2 - Q_3)^2}{2C_2} + \frac{Q_3^2}{2C_3}
\]

Inverse-Capacitance matrix:

\[
C_{ij}^{-1} = \frac{\partial^2 H}{\partial Q_i \partial Q_j}, \quad \Phi_i = \Phi_i^0 + \sum_j C_{ij}^{-1} Q_j
\]

Inverse-Inductance matrix:

\[
L_{ij}^{-1} = \frac{\partial^2 H}{\partial \Phi_i \partial \Phi_j}, \quad \dot{Q}_i = \frac{\partial}{\partial t} + \sum_j L_{ij}^{-1} \dot{\Phi}_j
\]

**COMMUTATION RELATION OF FLUXES AND CHARGES**

E.M. Lagrangian:

\[
\mathcal{L} = \frac{1}{2} \int_{\text{vol}} \left( e \epsilon E^2 - \mu B^2 \right) \, dv
\]

\[
= \int_{\text{vol}} \frac{1}{2} \left( e \epsilon \left( \frac{\partial}{\partial t} A \right)^2 - \mu \left( \nabla \times A \right)^2 \right) \, dv
\]

When magnetic and electric fields do not coexist in space (lumped elements), Hamiltonian of corresponding mode is given by:

\[
H = \frac{Q_{\text{cap}}^2}{2C} + \frac{\Phi_{\text{ind}}^2}{2L}
\]
CURRENTS AND VOLTAGES OF NORMAL MODES

Equation of motions in matrix form:

\[ V = \Phi = L^{-1} Q \]

\[ I = \dot{Q} = -L^{-1} \Phi \]

Matrix of eigenfrequencies

\[ \Omega = \left( L^{-1/2} C^{-1} L^{-1/2} \right)^{1/2} \]

\[ \Omega = \begin{pmatrix} \omega_a & 0 & 0 \\ 0 & \omega_b & 0 \\ 0 & 0 & \omega_c \end{pmatrix} \]

Impedance matrix

\[ Z = L^{-1/4} C^{-1/2} L^{1/4} \]

\[ Z = O^{-1} \begin{pmatrix} Z_a & 0 & 0 \\ 0 & Z_b & 0 \\ 0 & 0 & Z_c \end{pmatrix} O \]

\[ \Phi = \frac{hZ}{2} \left( a + a^\dagger \right) \]

\[ Q = \frac{h}{2Z} \left( a - a^\dagger \right) \]

1 photon

\[ @ f=10GHz \]

\[ Z = 100\Omega \]

\[ \delta V \sim 1\mu V \]

\[ \delta I \sim 10nA \]

AMPLIFICATION EXPLOITS COMBINATION OF ANHARMONICITY, DRIVE AND DAMPING

\[ H = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + \hbar \omega_c c^\dagger c + \left[ \hbar g^{(3)} a^\dagger b^\dagger c + \text{h.c.} \right] \]

\[ g^{(3)} \ll \omega_a, \omega_b, \omega_c \]

"3-wave mixing"

resonance condition:

\[ \omega_c = \omega_a + \omega_b \]

population inversion

\[ |001\rangle \]

pump

\[ |010\rangle \]

idler

\[ |110\rangle \]

signal

\[ |000\rangle \]
UNDERSTANDING OF AMPLIFICATION INVOLVES JOINT TREATMENT OF:

1. STEADY-STATE OUT-OF-EQUILIBRIUM

2. NON-LINEAR PROCESSES

OUTLINE

1. Modes of isolated, linear quantum circuits, non-linear processes

2. Open, out-of-equilibrium, linear systems: input-output theory

3. Characterizing non-linear elements, participation ratio
OPEN, DRIVEN, DISSIPATIVE LINEAR CIRCUITS

Couple circuit to out-of-equilibrium environment

Circuit is now excited

Environment is modeled as semi-infinite transmission lines carrying incoming signals

System loses energy by radiating outgoing signals into transmission lines, reaching steady state equilibrium.
**CORRESPONDANCE BETWEEN DRIVE AND INPUT FIELDS, DISSIPATION AND LINE IMPEDANCE**

Dissipative drive being equivalent to semi-infinite transmission line with incoming signals, system dynamics is described by relation between input and output fields.

\[
V(t) = V_{in} - RI(t) \quad \text{and} \quad I(t) = I_{in} - V(t)/R
\]

**Thévenin I**

\[
V_R = \frac{V_{in}}{\sqrt{R}} = \sqrt{R}I_{in}
\]

**Information carried by outgoing signal corresponds to that acquired by a voltmeter or an ammeter.**

**Hamiltonian of circuit** → functional relation between \( I \) and \( V \):

\[ f\{I(t),V(t)\} = 0 \]

Here, for the LC circuit, we obtain:

\[
\frac{d}{dt}I(t) = \left( C \frac{d^2}{dt^2} + \frac{1}{L} \right) V(t)
\]

In this relation, we now operate the substitution:

\[
V = \sqrt{R} \left[ A^* + A^{out} \right]
\]

\[
I = \frac{1}{\sqrt{R}} \left[ A^* - A^{out} \right]
\]

Expressing the outgoing field in terms of the incoming fields, we obtain the input-output relations.

For our example:

\[
C A^{out} + \frac{1}{R} A^{out} + \frac{1}{L} A^{out} = -C A^{in} + \frac{1}{R} A^{in} - \frac{1}{L} A^{in}
\]
FOURIER TRANSFORMS OF FIELDS

Starting from time-domain expression

\[ C \dot{A}^{\text{out}} + \frac{1}{R} \dot{A}^{\text{out}} + \frac{1}{L} A^{\text{out}} = -C \dot{A}^{\text{in}} + \frac{1}{R} \dot{A}^{\text{in}} - \frac{1}{L} A^{\text{in}} \]

and introducing Fourier transforms

\[ A^{\text{in/ou}}[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega t} A^{\text{in/ou}}(t) \, dt \]

We obtain:

\[ A^{\text{out}}[\omega] = \frac{-y(\omega) + 1}{y(\omega) + 1} A^{\text{in}}[\omega] \]

\[ y(\omega) = R\left(iC\omega + \frac{1}{iL\omega}\right) = \frac{\omega^2 - \omega_0^2}{i\gamma \omega} \]

When resonance quality factor \( Q = \frac{\omega_0}{\gamma} \gg 1 \) RWA, \( y(\omega) = e^{-\omega - \omega_0 \gamma / 2} \)

PHYSICAL MEANING OF DAMPING RATE IN INPUT-OUTPUT PICTURE

Propagating photons @ frequency \( \omega_0 \) enter the resonator and reside a time \( \gamma^{-1} \) as standing waves before being re-radiated back.

\[ \gamma = \frac{R_\text{a}}{Z_\text{a}} \omega_0 = R_\text{a} Y_\text{a} \omega_0 \]

phasors

\[ Z_c = R_\text{a} \]

resonator frequency \( \omega_0 \) presenting a \( || \) impedance \( Z_\text{a} \)

line for waves \( a \)

resonance \( Q \) determines sharpness

phase shift (deg)

0

180

360

\( \omega_0 \)

\( \gamma \)

11-II-15

11-II-16
SCATTERING MATRIX FOR A LINEAR SYSTEM

\[ \begin{bmatrix}
A_{11}[\omega] & A_{12}[\omega] & \cdots & \cdots & A_{1n}[\omega] \\
A_{21}[\omega] & A_{22}[\omega] & \cdots & \cdots & A_{2n}[\omega] \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
A_{11}[\omega] & A_{12}[\omega] & \cdots & \cdots & A_{1n}[\omega] \\
A_{21}[\omega] & A_{22}[\omega] & \cdots & \cdots & A_{2n}[\omega]
\end{bmatrix} =
\begin{bmatrix}
s_{11}[\omega] & s_{12}[\omega] & \cdots & \cdots & s_{1n}[\omega] \\
s_{21}[\omega] & s_{22}[\omega] & \cdots & \cdots & s_{2n}[\omega] \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
s_{11}[\omega] & s_{12}[\omega] & \cdots & \cdots & s_{1n}[\omega] \\
s_{21}[\omega] & s_{22}[\omega] & \cdots & \cdots & s_{2n}[\omega]
\end{bmatrix} \]

all components are at same frequency!
will change for amplifier

OUTGOING AMPLITUDES  "S" PARAMETERS, SCATTERING MATRIX  INCOMING AMPLITUDES

WAVE AMPLITUDE vs PHOTON OPERATORS

\[ A_{\text{in,out}}[\omega] = \sqrt{\frac{\hbar}{2}} a_{\text{in,out}}[\omega] \]

\[ \begin{bmatrix} A[\omega] \\ a[\omega] \end{bmatrix} = \begin{bmatrix} [\text{power}]^{1/2} \times \text{time} = [\text{action}]^{1/2} \\ [\text{time}]^{1/2} \end{bmatrix} \]

Ladder operators have commutation relations:

\[ [a_{\text{in,out}}[\omega_1], a_{\text{in,out}}[\omega_2]] = \text{sgn}(\omega_1 - \omega_2) \delta(\omega_1 + \omega_2) \]

Scattering always preserves commutation relations
**NUMBER OF PHOTONS IN INPUT SIGNAL**

Wave amplitude spectral density:

\[
\langle A[\omega_1] A[\omega_2] \rangle = S_{\omega \omega}[\omega] \delta(\omega_1 + \omega_2)
\]

In \( T \) equilibrium:

\[
S_{\omega \omega}[\omega] = \frac{\hbar \omega}{4} \left[ \coth \left( \frac{\hbar \omega}{2k_B T} \right) + 1 \right]
\]

\[
S_{\omega \omega}[\omega] \rightarrow \frac{k_B T}{2}
\]

Photon amplitude spectral density:

\[
\langle a[\omega_1] a[\omega_2] \rangle = S_{a a}[\omega] \delta(\omega_1 + \omega_2)
\]

Number of photons per unit time per unit bandwidth crossing a section of line:

\[
N_\omega[|\omega|] = S_{a a}[|\omega|] + S_{a a}[-|\omega|]
\]

In thermal equilibrium:

\[
\begin{align*}
N_\omega^+ (|\omega|) &= \frac{1}{2} \coth \left( \frac{\hbar |\omega|}{2k_B T} \right) \\
N_\omega^- (|\omega|) &\rightarrow \frac{k_B T}{2|\omega|} \\
N_\omega^0 (|\omega|) &\rightarrow \frac{1}{2}
\end{align*}
\]

N.B.: Link between "engineer" and "physicist" spectral densities

\[ S_{xx}[\nu] = S_{xx}[\omega = 2\pi \nu] + S_{xx}[\omega = -2\pi \nu] \]

**SCATTERING MATRIX FOR PASSIVE LINEAR CIRCUIT**

When circuit contains only linear capacitances and inductances:

\[
S[\omega] = \left( Z_c^{1/2} Y[\omega] Z_c^{1/2} + 1 \right)^{-1} \left( -Z_c^{1/2} Y[\omega] Z_c^{1/2} + 1 \right)
\]

Admittance matrix of circuit

Gives current in port \( k \) as function of voltages in port \( l \). Can be computed directly from hamiltonian.

Diagonal matrix of line impedances

This expression is a generalization of the formula for reflection on a load:

\[
r = \frac{Z_L - Z_c}{Z_L + Z_c}
\]

\( Y \) matrix is \( i \) times a positive hermitian matrix

unitarity of \( S \) matrix

\( S^\dagger S = 1 \)

\( \{ \text{information conservation} \}

\{ \text{energy conservation} \}

Another property, fully general:

causality

\[ \{ \text{poles of } S \text{ matrix} \}

\text{in lower-half plane} \]
EXAMPLE: INPUT-OUTPUT TREATMENT OF DISPERSIVE REFLECTION ATTENUATOR

\[ \gamma_a, \gamma_b \]

waves \( a \) \( Z_c = R_a \)
\( \omega_0 \)
\( 2C \)
\( 2C \)
\( Z_c = R_b \)
waves \( b \)

\[ C_{a,b} \ll C \]

\[ s_{aa}(\omega) \]

attenuation of reflected signal

\[ S = \begin{bmatrix} \frac{\Gamma_a - \Gamma_b - i\Delta\omega}{\Gamma_a + \Gamma_b + i\Delta\omega} & \frac{2\Gamma_a^{1/2}\Gamma_b^{1/2}}{\Gamma_a + \Gamma_b + i\Delta\omega} \\ \frac{2\Gamma_a^{1/2}\Gamma_b^{1/2}}{\Gamma_a + \Gamma_b + i\Delta\omega} & \frac{\Gamma_b - \Gamma_a - i\Delta\omega}{\Gamma_a + \Gamma_b + i\Delta\omega} \end{bmatrix} \]

\[ \Delta\omega = \omega - \omega_0 \]

Interference: when \( \Delta\omega = 0 \) & \( \gamma_a = \gamma_b \), reflection vanishes!

Added noise by attenuator in reflection of \( a \):

\[ A = N_b \left( \frac{|s_{ab}|}{|s_{aa}|} \right)^2 = N_b \left( |s_{aa}|^{-2} - 1 \right) \]

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**JE = NON-DISSIPATIVE, NON-LINEAR INDUCTANCE**

\[ V(t) = L \frac{dI(t)}{dt} \]

\[ V(t) = \phi_0 \frac{d}{dt} \sin^{-1} \left( \frac{I(t)}{I_0} \right) \]

\[ \phi_0 = \frac{\hbar}{2e} \]

Energy stored in Josephson element:

\[ E(t) = \int_0^{I(t)} V(t) dI = E_J \left( 1 - \cos \left[ \sin^{-1} \left( \frac{I(t)}{I_0} \right) \right] \right) \]

\[ E_J = \phi_0 I_0 \]

**CHARACTERIZING NON-LINEARITY**

\[ L_j = \frac{\hbar^2}{(2e)^2} E_j \]

\[ \delta L_{JNL} = L_j \left[ \left( 1 - \frac{I^2}{I_0^2} \right)^{-1/2} - 1 \right] \]

\[ = \frac{1}{2} L_j \left( \frac{I}{I_0} \right)^2 + O \left( \frac{I}{I_0} \right) \]

\[ \omega(E) = \frac{1}{\sqrt{L(E)C}} \]

Anharmonicity

| classical | \( \frac{\partial \omega}{\partial E} \) | \( \frac{E_{\text{max}}}{\omega(0)} = 1/4 \) |
| quantum  | \( \frac{\partial \omega}{\partial E} \) | \( \hbar \omega(0) / \gamma \) |

\[ \omega(0) = \omega(0)(1 - E/8E_j) \]

\[ \omega(0) \]

\[ U \]

\[ E_{\text{max}} = 2E_j \]
**SQUID's: MODULATION OF NON-LINEARITY**

\[ L(I) = \frac{\Phi_0}{I_{\text{eff}}} \left[ 1 + \frac{1}{2} \frac{I^2}{I_{\text{eff}}^2} + O\left(\frac{I^4}{I_{\text{eff}}^4}\right) \right] \]

\[ I_{\text{eff}} = 2I_0 \cos\left(\frac{\Phi_{\text{ext}}}{2\Phi_0}\right) \]

non-linearity maximum here

see Benjamin Huard's seminar

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**EVALUATION OF \( g^{(3)} \)**

- \( I_0 \approx 10\mu\text{A} \quad \Rightarrow \quad L_J \approx 30\text{pH} \)
- \( E_J/h \approx 5\text{THz} \)
- \( f \approx 5\text{GHz}, Z \approx 1\Omega \)

1 photon \( \approx 100\text{nA} \)

\( \sim I_0/100 \)

Non-linear term in Hamiltonian is of order:

\[ \frac{E_J}{(I_0)^3} \Rightarrow g^{(3)} \sim \text{MHz} \]
DILUTION OF NON-LINEARITY

\[ L(I) = J_L \left[ 1 + \frac{1}{2} \frac{I^2}{I_0^2} + O\left(\frac{I^4}{I_0^4}\right) \right] \]

\[ L(I) = L_s + L_j \]

\[ p = \frac{L_j}{L_s + L_j} \]

decrease in participation ratio

BANDWIDTH OF NON-LINEARITY

\[ \omega_p = \sqrt{\frac{1}{L_j C_j}} \]

Plasma frequency is ultimate bandwidth limitation (20-30GHz)

Nanobridges have a higher plasma frequency than tunnel junctions and are promising non-linear elements for superconducting amplifiers.

SELECTED BIBLIOGRAPHY

Books and series of lectures
Braginsky, V. B., and F. Y. Khalili, “Quantum Measurements” (Cambridge University Press, Cambridge, 1992)
Clarke, J. and Braginsky, A. I., eds., "The SQUID Handbook" (Wiley-VCH, Weinheim, Germany, 2006)
Haroche, S., Lectures at College de France, 2011
Haroche, S. and Raimond, J-M., "Exploring the Quantum" (Oxford University Press, 2006)
Nielsen, M. and Chuang, I., “Quantum Information and Quantum Computation” (Cambridge, 2001)
Walls, D.F., and Milburn, G.J. "Quantum Optics" (Springer, Berlin, 2008)
Wiseman, H. M. and Milburn, G. J., "Quantum Measurement and Control" (Cambridge, 2011)

Articles