COLLÈGE DE FRANCE 1530

Chaire de Physique Mésoscopique
Michel Devoret
Année 2011, 10 mai - 21 juin

AMPLIFICATION ET RETROACTION QUANTIQUES

QUANTUM AMPLIFICATION AND FEEDBACK

Troisième Leçon / *Third Lecture*

Transparents des leçons disponibles à http://www.physinfo.fr/lectures.html

11-III-1

PROGRAM OF THIS YEAR'S LECTURES

Lecture I: Introduction to quantum-limited amplification and feedback

Lecture II: How do we model open, out-of-equilibrium, non-linear quantum systems?

Lecture III: Can we maintain the noise at the quantum limit while increasing gain, bandwidth and dynamic range?

Lecture IV: What are the minimal requirements for an active circuit to be fully directional and noiseless?

Lecture V: Can continuous quantum measurements be viewed as a form of Brownian motion?

Lecture VI: How can we maintain a dynamic quantum state alive?

11-III-2

CALENDAR OF SEMINARS

May 10: Fabien Portier, SPEC-CEA Saclay
The Bright Side of Coulomb Blockade

May 17, 2011: Jan van Ruitenbeek (Leiden University, The Netherlands)
Quantum Transport in Single-molecule Systems

May 31, 2011: Irfan Siddiqi (UC Berkeley, USA)
Quantum Jumps of a Superconducting Artificial Atom

June 7, 2011: David DiVicenzo (IQI Aachen, Germany)
Quantum Error Correction and the Future of Solid State Qubits

June 14, 2011: Andrew Cleland (UC Santa Barbara, USA)
Images of Quantum Light

June 21, 2011: Benjamin Huard (LPA - ENS Paris)
Building a Quantum Limited Amplifier from Josephson Junctions and Resonators

June 21, 2011 (3pm): Andrew Cleland (UC Santa Barbara, USA)
How to Be in Two Places at the Same Time ?

11-III-3

LECTURE III : CALCULATING THE PERFORMANCES OF A QUANTUM-LIMITED AMPLIFIER

OUTLINE

1. Anatomy of the amplifier; list of parts
2. The amplifier differential equations: quantum and stochastic
3. Compromise between gain, bandwidth and dynamic range

11-III-4

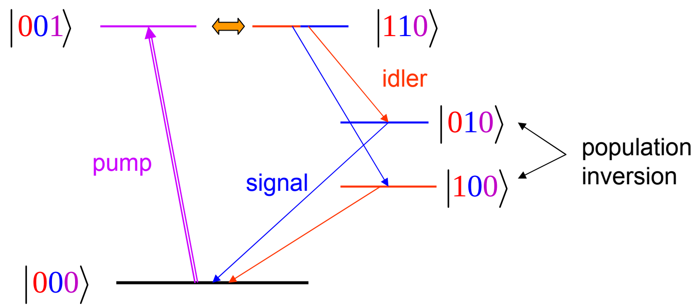
AMPLIFICATION EXPLOITS COMBINATION OF ANHARMONICITY, DRIVE AND DAMPING

$$H_{rev} = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\omega_c c^\dagger c + \left[\hbar g^{(3)} a^\dagger b^\dagger c + \text{h.c.}\right]$$

$$g^{(3)} \ll \omega_a, \omega_b, \omega_c$$

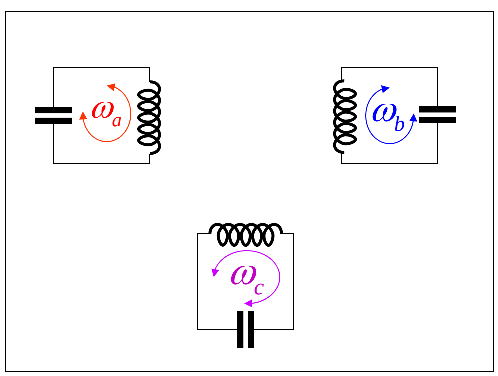
resonance condition:

$$\omega_c = \omega_a + \omega_b$$



11-II-10bis

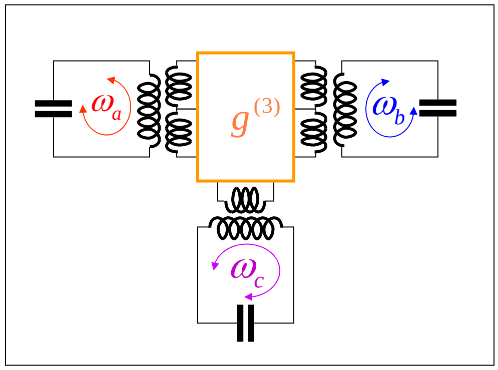
THE LIST OF PARTS(1)



$$\frac{H_{rev}^{lin}}{\hbar} = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_c c^\dagger c$$

11-III-5

THE LIST OF PARTS(1)

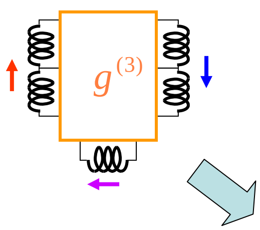


$$H_{rev} = H_{rev}^{lin} + H_{rev}^{nl}$$

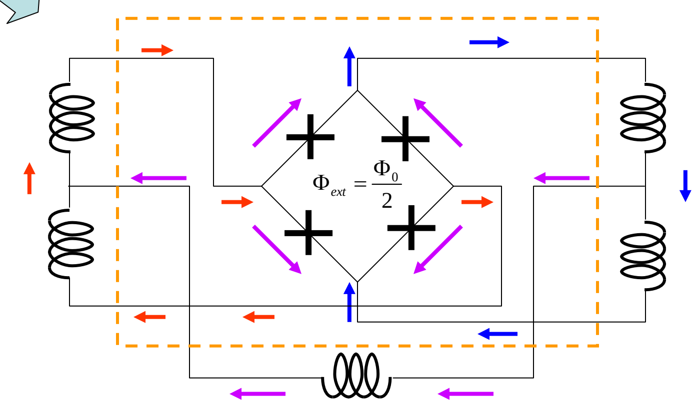
$$\frac{H_{rev}^{lin}}{\hbar} = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_c c^\dagger c$$

$$\frac{H_{rev}^{nl}}{\hbar} = g^{(3)} a^\dagger b^\dagger c + \text{h.c.}$$

11-III-5a

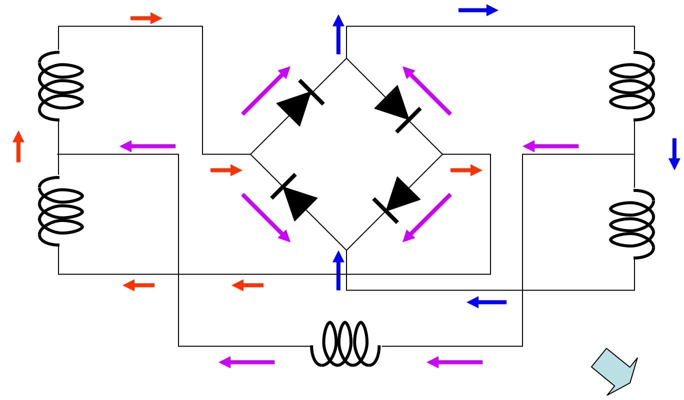


THE PURELY INDUCTIVE MIXER

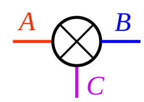


11-III-6

RING MODULATOR PERFORMS PRODUCT OF TWO SIGNALS

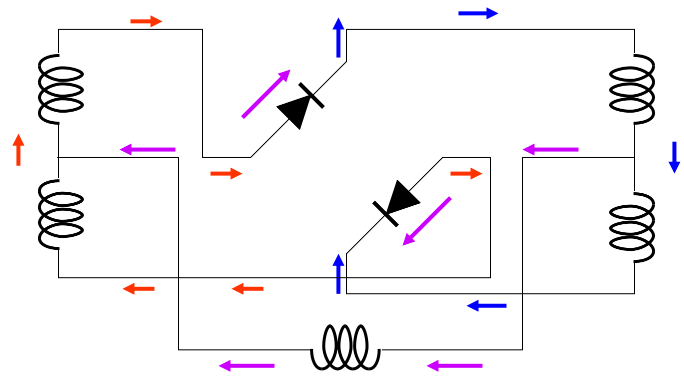


$B^{out} \sim A^{in} \times C^{in}$; $A^{out} \sim B^{in} \times C^{in}$; $C^{out} \sim A^{in} \times B^{in}$



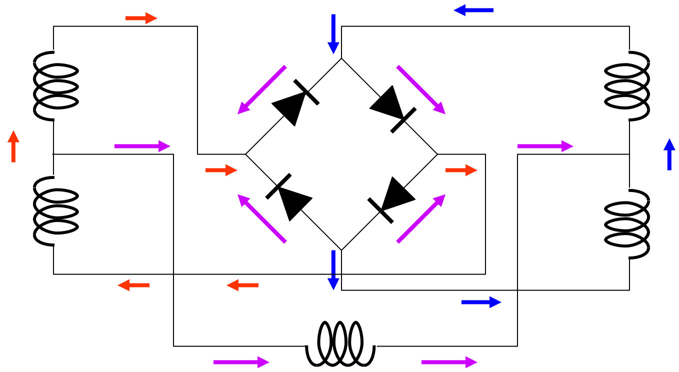
11-III-7

RING MODULATOR PERFORMS PRODUCT OF TWO SIGNALS



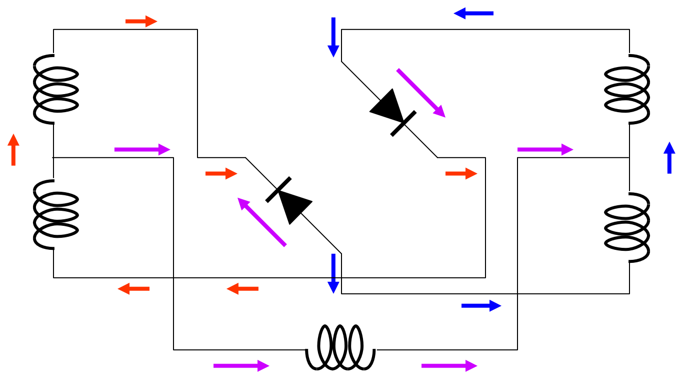
11-III-7a

RING MODULATOR PERFORMS PRODUCT OF TWO SIGNALS



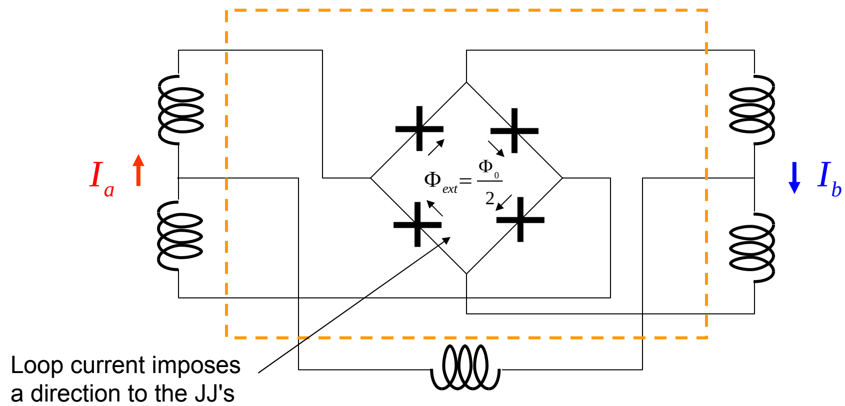
11-III-7b

RING MODULATOR PERFORMS PRODUCT OF TWO SIGNALS



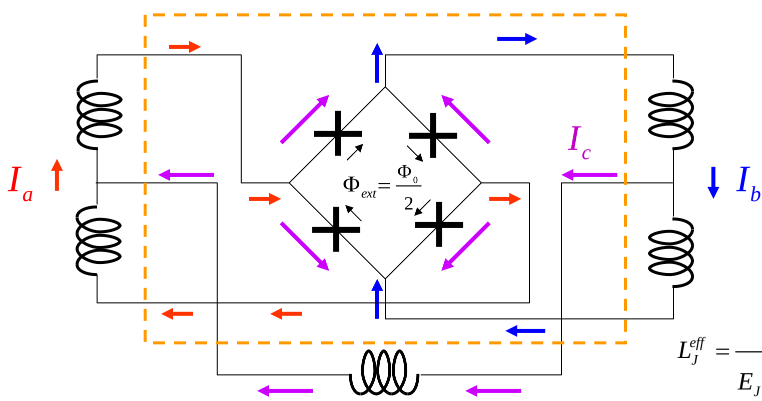
11-III-7c

JOSEPHSON MODULATOR: REVERSIBLE MIXING



11-III-8

JOSEPHSON MODULATOR: REVERSIBLE MIXING



$$L_J^{eff} = \frac{\varphi_0^2}{E_J \cos\frac{\Phi_{ext}}{\varphi_0}}$$

$$\varphi_0 = \frac{\hbar}{2e}$$

$$E_{ring} = \frac{L_J^{eff}}{2}\left(I_a^2 + I_b^2\right) + \frac{L_J^{eff}}{8} I_c^2 + \lambda I_a I_b I_c + \text{higher order terms}$$

$$\lambda = \frac{\left(L_J^{eff}\right)^2}{4\varphi_0}$$

Wheatstone bridge symmetry eliminates many undesirable terms:
- ~~$I_a I_b$~~
- ~~I_a^3~~
- ~~$I_a^2 I_b$~~

11-III-8 a

FROM $\lambda I_a I_b I_c$ TO $g^{(3)} a^\dagger b^\dagger c$

$$I_a = \omega_a \sqrt{\frac{\hbar}{2Z_a}}\left(a + a^\dagger\right)$$

$$I_b = \omega_b \sqrt{\frac{\hbar}{2Z_b}}\left(b + b^\dagger\right)$$

$$I_c = \omega_c \sqrt{\frac{\hbar}{2Z_c}}\left(c + c^\dagger\right)$$

$$g^{(3)} = \frac{1}{8}\sqrt{\frac{\left(L_J^{eff}\right)^4 \omega_a^2 \omega_b^2 \omega_c^2}{2 Z_a Z_b Z_c}\frac{(2e)^2}{\hbar}}$$

$$= \frac{1}{8}\sqrt{\frac{\left(L_J^{eff}\right)^3 \omega_a \omega_b \omega_c}{2 L_a L_b L_c \omega_J^{eff}}}$$

$$\sim \mathrm{MHz}$$

$$\omega_J^{eff} = \frac{I_0^{eff}}{2e}$$

ATTENTION: LIMITS TO PHOTON NUMBERS!

$$n_{a,b,c} \ll \frac{\omega_J^{eff\,2}}{\omega_{a,b,c}^{\;2}} \frac{Z_{a,b,c}}{R_Q} \sim 10^4$$

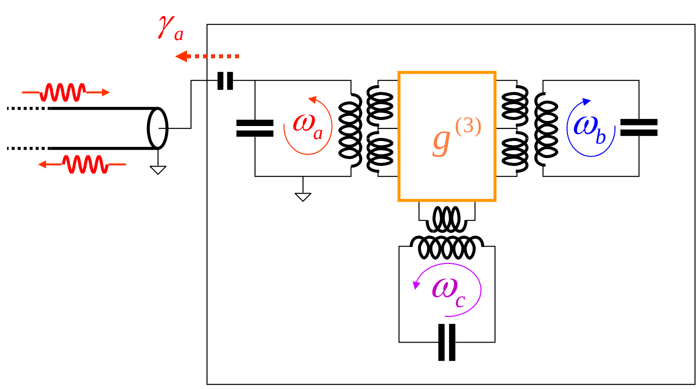
present technology

$$R_Q = \frac{\hbar}{(2e)^2} \simeq 1\,\mathrm{k}\Omega$$

11-III-9

THE LIST OF PARTS(2)

$$H = H_{rev} + H_{coupl} + H_{env}$$



$$H_{rev} = H_{rev}^{lin} + H_{rev}^{nl}$$

$$\frac{H_{rev}^{lin}}{\hbar} = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_c c^\dagger c$$

$$\frac{H_{rev}^{nl}}{\hbar} = g^{(3)} a^\dagger b^\dagger c + \mathrm{h.c.}$$

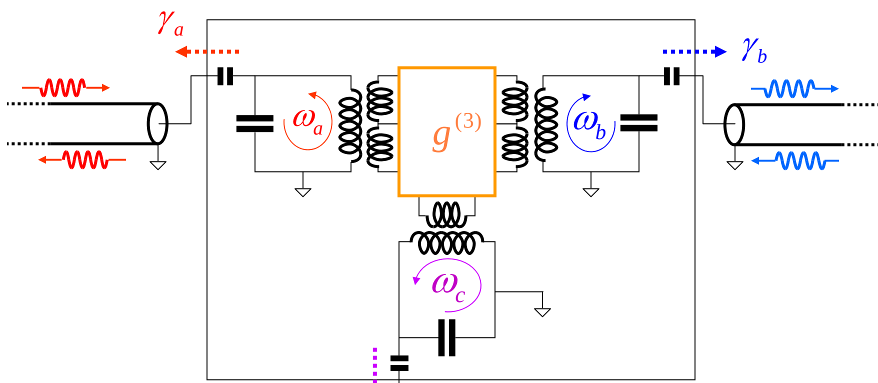
$$H_{coupl} = \sqrt{\frac{\gamma_a}{\pi}}\left(a + a^\dagger\right)\int_{-\infty}^{+\infty} d\omega \frac{A[-\omega]}{\sqrt{\hbar|\omega|}}$$

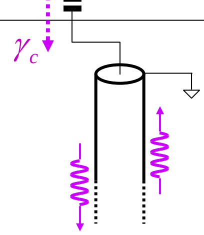
$$H_{env} = \int_{-\infty}^{+\infty} d\omega A[-\omega] A[\omega]$$

11-III-10

THE LIST OF PARTS(2)

$$H = H_{rev} + H_{coupl} + H_{env}$$





$$H_{rev} = H_{rev}^{lin} + H_{rev}^{nl}$$

$$\frac{H_{rev}^{lin}}{\hbar} = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_c c^\dagger c$$

$$\frac{H_{rev}^{nl}}{\hbar} = g^{(3)} a^\dagger b^\dagger c + \text{h.c.}$$

$$H_{coupl} = \sqrt{\frac{\gamma_a}{\pi}}\left(a + a^\dagger\right)\int_{-\infty}^{+\infty} d\omega \frac{A[-\omega]}{\sqrt{\hbar|\omega|}}$$

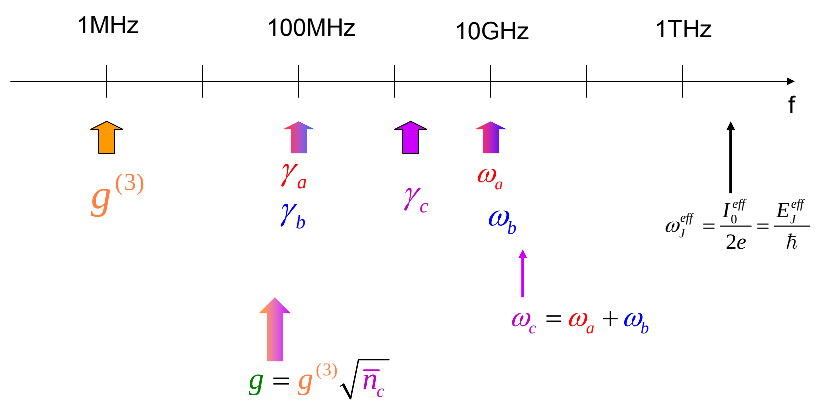
$+ bB$ term $\quad + cC$ term

$$H_{env} = \int_{-\infty}^{+\infty} d\omega A[-\omega] A[\omega]$$

$+ BB$ term $\quad + CC$ term

11-III-10a

CHARACTERISTIC FREQUENCIES AND ENERGY SCALES



Staying below critical photon number: $\sqrt{p_a p_b Q_a Q_b} > 4$

participation ratio $p_i = L_J^{eff}(L_i + L_J^{eff})$ quality factor ω_i / γ_i

11-III-11

VARIATIONS ON AMPLIFIERS

Non-degenerate: $\hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\omega_c c^\dagger c + \left[\hbar g^{(3)} a^\dagger b^\dagger c + \mathrm{h.c.}\right]$

$\omega_c = \omega_a + \omega_b$ (Yale, Paris)

Degenerate: $\hbar\omega_a a^\dagger a + \hbar\omega_c c^\dagger c + \left[\hbar g^{(3)} a^{\dagger 2} c + \mathrm{h.c.}\right]$

$\omega_c = 2\omega_a$ (Tsukuba, Göteborg)

Doubly-degenerate: $\hbar\omega_a a^\dagger a + \hbar\omega_c c^\dagger c + \left[\hbar g^{(4)} a^{\dagger 2} c^2 + \mathrm{h.c.}\right]$

$\omega_c = \omega_a$

(Boulder, Berkeley)

Can be used also as bifurcation amplifier (Yale, Saclay, Berkeley,...)

11-III-12

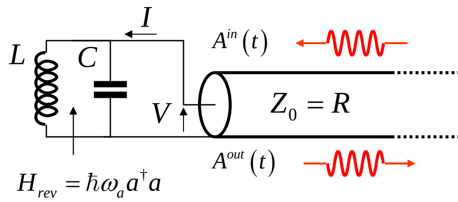
LECTURE III : PREDICTING THE PERFORMANCES OF A QUANTUM-LIMITED AMPLIFIER

<u>OUTLINE</u>

1. Anatomy of the amplifier; list of parts
2. The amplifier differential equations: quantum and stochastic
3. Compromise between gain, bandwidth and dynamic range

11-III-4

INPUT-OUTPUT EQUATIONS IN PHOTON AMPLITUDE LANGUAGE



Circuit equations with input field in standard form:

$$\left(C\frac{d^2}{dt^2}+\frac{1}{L}\right)V(t)=\frac{d}{dt}I(t)=\frac{2}{\sqrt{R}}\frac{d}{dt}A^{in}(t)-\frac{d}{dt}\frac{V(t)}{R}$$

$$V=\sqrt{R}\left[A^{in}+A^{out}\right]$$

$$I=\frac{1}{\sqrt{R}}\left[A^{in}-A^{out}\right]$$

Voltage → photon amplitude: $V(t)=\frac{d}{dt}\sqrt{\frac{\hbar Z_a}{2}}\left(a(t)+a^\dagger(t)\right)$ $\quad \omega_a=\sqrt{\frac{1}{LC}};\ Z_a=\sqrt{\frac{L}{C}}$

$$A^{in}[\omega]=\sqrt{\frac{\hbar|\omega|}{2}}a^{in}[\omega] \qquad \left[a^{in}[\omega_1],a^{in}[\omega_2]\right]=\mathrm{sgn}(\omega_1-\omega_2)\,\delta(\omega_1+\omega_2)$$

Circuit equations with input field in photon amplitude form:

$$\left(\frac{d^2}{dt^2}+\gamma_a\frac{d}{dt}+\omega_a^2\right)\left(a+a^\dagger\right)(t)=2\omega_a\sqrt{\frac{2\gamma_a}{\hbar\omega_a}}A^{in}(t)=2\omega_a\int_{-\infty}^{+\infty}d\omega\sqrt{\frac{\gamma_a|\omega|}{\omega_a}}\mathrm{e}^{-i\omega t}a^{in}[\omega]$$

PHOTON AMPLITUDE INPUT-OUTPUT RELATIONS IN ROTATING WAVE APPROXIMATION

Last equation, in Fourier domain: $\left(\omega_a^2-\omega^2-i\gamma_a\right)a[\omega]=2\omega_a\sqrt{\frac{\gamma_a|\omega|}{\omega_a}}a^{in}[\omega]$

Validity condition of RWA:

$$\gamma_a=2\Gamma_a\ll\omega_a \qquad a[\omega]\cong\frac{2\omega_a}{\left[\left(\omega_a-i\Gamma_a\right)^2-\omega^2\right]}\sqrt{\frac{\gamma_a|\omega|}{\omega_a}}a^{in}[\omega]\cong\frac{\sqrt{\gamma_a}a^{in}[\omega]}{\omega_a-i\Gamma_a-\omega}$$

Back in the time domain: $\frac{d}{dt}a=-i\omega_a a-\frac{\gamma_a}{2}a+\sqrt{\gamma_a}a^{in}$ (have dropped unimportant phase factor in input wave)

More generally: $\frac{d}{dt}a=\frac{i}{\hbar}\left[H_{rev},a\right]-\frac{\gamma_a}{2}a+\sqrt{\gamma_a}a^{in}$ Quantum Langevin Equation

REMARKS ON THE QUANTUM LANGEVIN EQUATION

$$\frac{d}{dt}a = \frac{i}{\hbar}[H_{rev}, a] - \frac{\gamma_a}{2}a + \sqrt{\gamma_a}\,a^{in}$$

Operator equation extending Heisenberg' equation

Contains damping + drive + noise

Written here in RWA for single field and single operator,
but can be more refined

Implements Fluctuation Dissipation Theorem

Easily transformed to form giving input-output relations:

$$\sqrt{\gamma_a}\,a = a^{in} + a^{iout} \quad \Rightarrow \quad \left(\frac{d}{dt} + i\omega_a + \frac{\gamma_a}{2}\right)a^{out} = \left(-\frac{d}{dt} - i\omega_a + \frac{\gamma_a}{2}\right)a^{in}$$

11-III-14bis

AMPLIFIER EQUATIONS

$$\frac{d}{dt}a = -i\omega_a a - i g^{(3)} b^{\dagger} c - \frac{\gamma_a}{2}a + \sqrt{\gamma_a}\,a^{in}$$

$$\frac{d}{dt}b = -i\omega_b b - i g^{(3)} a^{\dagger} c - \frac{\gamma_b}{2}b + \sqrt{\gamma_b}\,b^{in}$$

$$\frac{d}{dt}c = -i\omega_c c - i g^{(3)*} ab - \frac{\gamma_c}{2}c + \sqrt{\gamma_c}\,c^{in}$$

Crucial remark: Quantum fluctuations in the c field are negligible compared with deterministic amplitude determined by the pump.

$$c(t) = \langle c(t)\rangle + \delta c(t)$$

$$\langle c(t)\rangle = \sqrt{\bar{n}_c}\,e^{-i\omega_c t}$$

$$\langle (\delta c)^2 \rangle \ll \langle c \rangle^2$$

Input field = coherent state (Glauber)

$$\left|\Psi_c^{in}(t)\right\rangle = \mathrm{e}^{-\bar{n}_c}\sum_{n_c}\frac{(\bar{n}_c)^{n_c}\,\mathrm{e}^{-in_c\omega_c t}}{\sqrt{n_c\,!}}\left|n_c\right\rangle$$

11-III-15

FLUCTUATION-LESS PUMP APPROXIMATION

$$c(t) \to \langle c(t) \rangle = \sqrt{\bar{n}_c}\, e^{-i(\omega_c t + \phi)} \qquad g = g^{(3)} \sqrt{\bar{n}_c}\, e^{-i\phi} \qquad \Gamma_{a,b} = \frac{\gamma_{a,b}}{2}$$

$$\begin{cases} \left(\frac{d}{dt} + i\omega_a + \Gamma_a\right) a^{out} + ig\sqrt{\frac{\Gamma_a}{\Gamma_b}} e^{-i\omega_c t} b^{\dagger out} = \left(-\frac{d}{dt} - i\omega_a + \Gamma_a\right) a^{in} - ig\sqrt{\frac{\Gamma_a}{\Gamma_b}} e^{-i\omega_c t} b^{\dagger in} \\ \left(\frac{d}{dt} + i\omega_b + \Gamma_b\right) b^{out} + ig\sqrt{\frac{\Gamma_b}{\Gamma_a}} e^{-i\omega_c t} a^{\dagger out} = \left(-\frac{d}{dt} - i\omega_b + \Gamma_b\right) b^{in} - ig\sqrt{\frac{\Gamma_b}{\Gamma_a}} e^{-i\omega_c t} a^{\dagger in} \end{cases}$$

After a Fourier transform, in frequency domain:

$$\begin{bmatrix} h_a[\omega_s] & ig_b^a \\ -ig_a^{b*} & h_b^*[\omega_I] \end{bmatrix} \begin{bmatrix} a^{out}[+\omega_s] \\ b^{out}[-\omega_I] \end{bmatrix} = \begin{bmatrix} h_a^*[\omega_s] & -ig_b^a \\ +ig_a^{b*} & h_b[\omega_I] \end{bmatrix} \begin{bmatrix} a^{in}[+\omega_s] \\ b^{in}[-\omega_I] \end{bmatrix}$$

Where: $\omega_S + \omega_I = \omega_c \qquad h_{a,b}[\omega] = -i\omega + i\omega_{a,b} + \Gamma_{a,b}$

11-III-16

SCATTERING MATRIX

$$\begin{bmatrix} a^{out}[+\omega_s] \\ b^{out}[-\omega_I] \end{bmatrix} = \begin{bmatrix} r_{aa} & s_{ab} \\ s_{ba} & r_{bb} \end{bmatrix} \begin{bmatrix} a^{in}[+\omega_s] \\ b^{in}[-\omega_I] \end{bmatrix}$$

$$r_{aa} = \frac{\eta_a^* \eta_b^* + |\rho|^2}{\eta_a \eta_b^* - |\rho|^2} \qquad r_{bb} = \frac{\eta_a \eta_b + |\rho|^2}{\eta_a \eta_b^* - |\rho|^2} \qquad s_{ab} = \frac{-2i\rho}{\eta_a \eta_b^* - |\rho|^2} \qquad s_{ba} = \frac{2i\rho^*}{\eta_a \eta_b^* - |\rho|^2}$$

$$\eta_a = 1 - i\frac{\omega_S - \omega_a}{\Gamma_a} \qquad \eta_b = 1 - i\frac{\omega_I - \omega_b}{\Gamma_b} \qquad \rho = \frac{g^{(3)}\sqrt{\bar{n}_c}\, e^{-i\phi}}{\sqrt{\Gamma_a \Gamma_b}}$$

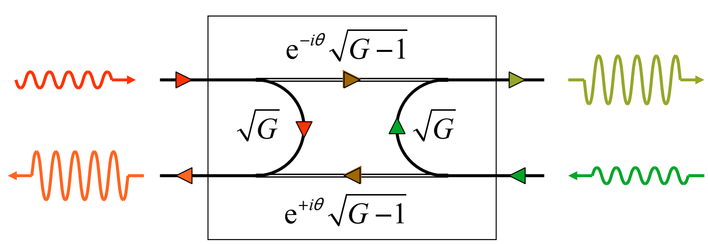
For zero detuning, i.e. $\eta_a = \eta_b = 1 \Rightarrow r_{aa} = r_{bb} = \sqrt{G_0}$

$$G_0 = \left(\frac{1 + |\rho|^2}{1 - |\rho|^2}\right)^2 = \left(\frac{\Gamma_a + \Gamma_b^{\text{amp}}}{\Gamma_a - \Gamma_b^{\text{amp}}}\right)^2 \qquad \Gamma_b^{\text{amp}} = \frac{|g^{(3)}|^2 \bar{n}_c}{\Gamma_b}$$

Idler port appears as negative impedance port to the signal!!

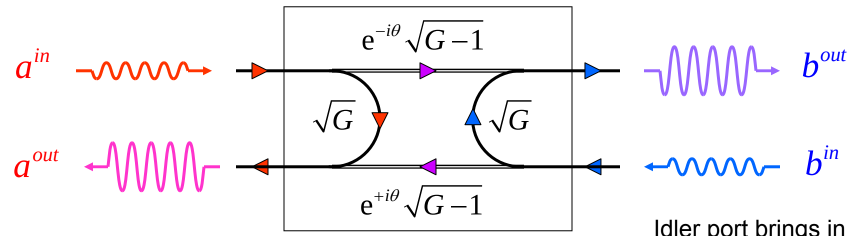
formula for attenuator, with a Γ<0 for aux. port (see 2nd lecture)

11-III-17



$e^{-i\theta}\sqrt{G-1}$
$\sqrt{G}$
$\sqrt{G}$
$e^{+i\theta}\sqrt{G-1}$

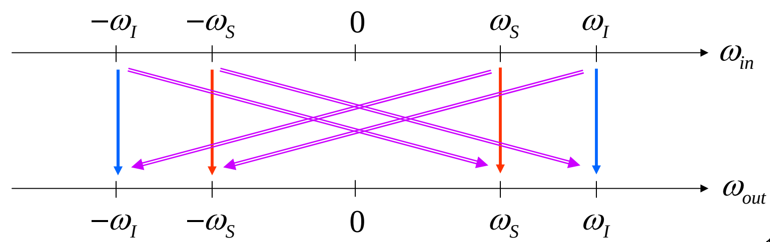
PHASE CONJUGATION



Reflection amplifier

Idler port brings in minimum added noise

$$a^{out} = \sqrt{G}\,a^{in} + e^{i\theta}\sqrt{G-1}\,b^{in\dagger}$$



11-III-18a

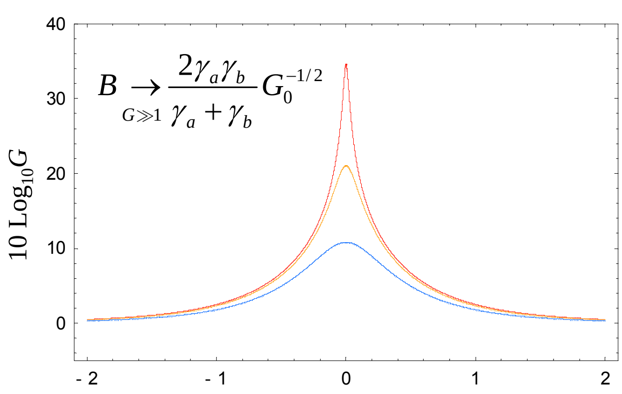
BANDWIDTH

$$\Delta\omega = \omega_S - \omega_a = \omega_b - \omega_I$$

$$G(\Delta\omega) \underset{|\rho|\to 1^-}{\rightarrow} \left| \frac{1 - i\Delta\omega\left(\Gamma_a^{-1} - \Gamma_b^{-1}\right) + |\rho|^2}{1 + i\Delta\omega\left(\Gamma_a^{-1} + \Gamma_b^{-1}\right) - |\rho|^2} \right|^2 \underset{|\rho|\to 1^-}{\rightarrow} \frac{G_0}{1 + \left(\dfrac{\Delta\omega}{\gamma G_0^{-1/2}}\right)^2}$$

Bandwidth decreases as gain increases:

$$\gamma^{-1} = \gamma_a^{-1} + \gamma_b^{-1}$$

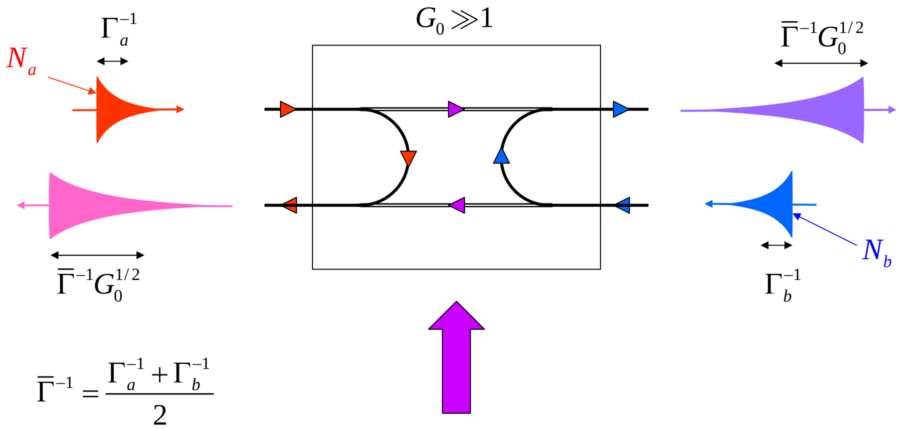


$\longrightarrow \dfrac{\Delta\omega}{2\gamma}$

General result valid for all parametric amplifiers

11-III-19

ENERGY BUDGET



$$\bar{\Gamma}^{-1} = \frac{\Gamma_a^{-1} + \Gamma_b^{-1}}{2}$$

$$N_c = \bar{\Gamma}^{-1} G_0^{1/2} \left(\Gamma_a + \Gamma_b\right)\left(N_a + N_b\right)$$

SELF-CONSISTENT PUMP EQUATIONS

$$\frac{d}{dt}\langle c \rangle = -i\omega_c \langle c \rangle - i g^{(3)*} \langle ab \rangle - \frac{\gamma_c}{2}\langle c \rangle + \sqrt{\gamma_c}\langle c^{in} \rangle$$

Correlation between the *a* and *b* waves: $-i g^{(3)} \langle ab \rangle = -\frac{\gamma^{eff}(G)}{2}\langle c \rangle$

where $\gamma^{eff}(G) = \frac{1}{\pi}\frac{\left|g^{(3)}\right|^2}{\Gamma_a \Gamma_b}\int_{-\infty}^{+\infty} d(\Delta\omega)\left(N_a(\Delta\omega) + N_b(\Delta\omega)\right) G(\Delta\omega)$

For noise:

$$\gamma^{eff}(G) = \frac{2\left|g^{(3)}\right|^2 \gamma G_0^{1/2}}{\Gamma_a \Gamma_b}\left(N_a + N_b\right)$$

For CW power:

$$\gamma^{eff}(G) = \frac{\left|g^{(3)}\right|^2 G_0}{\pi \Gamma_a \Gamma_b}\left(P_a + P_b\right)$$

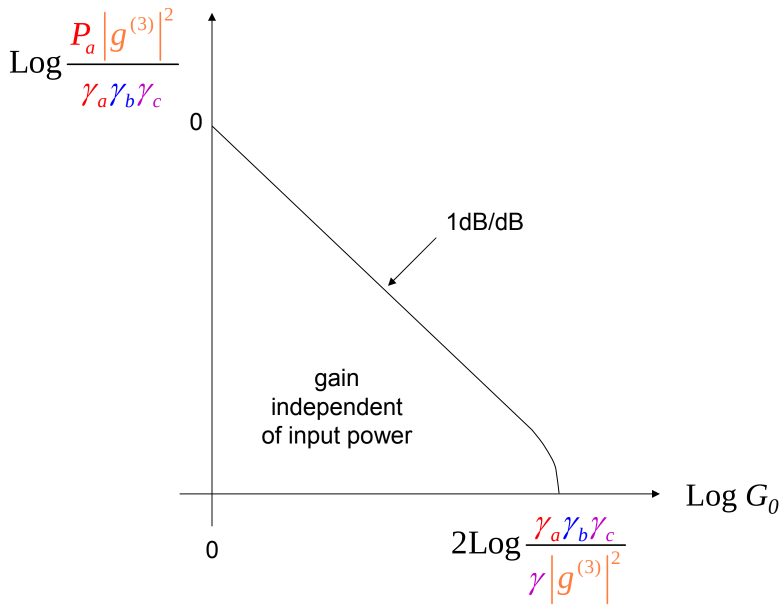
$$\bar{n}_c = \frac{4\gamma_c}{\gamma_c^2 + \gamma_c^{eff\,2}}\bar{n}_c^{in}$$

Pump depletion negligible if

$$\gamma_c^{eff} \ll \gamma_c$$

DETERMINES DYNAMIC RANGE

DYNAMIC RANGE LIMITATIONS



11-III-22

END OF LECTURE