



COLLÈGE
DE FRANCE
—1530—



Chaire de Physique Mésoscopique
Michel Devoret
Année 2012, 15 mai - 19 juin

RÉSONATEURS NANOMÉCANIQUES DANS LE RÉGIME QUANTIQUE

NANOMECHANICAL RESONATORS IN QUANTUM REGIME

Cinquième leçon / *Fifth lecture*

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PROGRAM OF THIS YEAR'S LECTURES

Lecture I: Introduction to nanomechanical systems

Lecture II: How do we model the coupling between electromagnetic modes and mechanical motion?

Lecture III: Is zero-point motion of the detector variables a limitation in the measurement of the position of a mechanical oscillator?

Lecture IV: In the active cooling of a nanomechanical resonator, is measurement-based feedback equivalent to autonomous feedback?

Lecture V: How close to the ground state can we bring a nanoresonator?

Lecture VI: What oscillator characteristics must we choose to convert quantum information from the microwave domain to the optical domain?

CALENDAR OF 2012 SEMINARS

May 15: Rob Schoelkopf (Yale University, USA)

Quantum optics and quantum computation with superconducting circuits.

May 22: Konrad Lehnert (JILA, Boulder, USA)

Micro-electromechanics: a new quantum technology.

May 29: Olivier Arcizet (Institut Néel, Grenoble)

A single NV defect coupled to a nanomechanical oscillator: hybrid nanomechanics.

June 5: Ivan Favero (MPQ, Université Paris Diderot)

From micro to nano-optomechanical systems: photons interacting with mechanical resonators.

June 12: A. Douglas Stone (Yale University, USA)

Lasers and anti-lasers: a mesoscopic physicist's perspective on scattering from active and passive media.

June 19: Tobias J. Kippenberg (EPFL, Lausanne, Suisse)

Cavity optomechanics: exploring the coupling of light and micro- and nanomechanical oscillators.

LECTURE V : INPUT-OUTPUT TREATMENT OF COLD DAMPING IN QUANTUM REGIME

OUTLINE

1. Qualitative discussion of lowest temperatures achievable by dynamical cooling.
2. Review of input-output theory of driven dissipative systems in quantum regime.
3. Photon-phonon inelastic scattering between opto- and electro-mechanical nanoresonators.

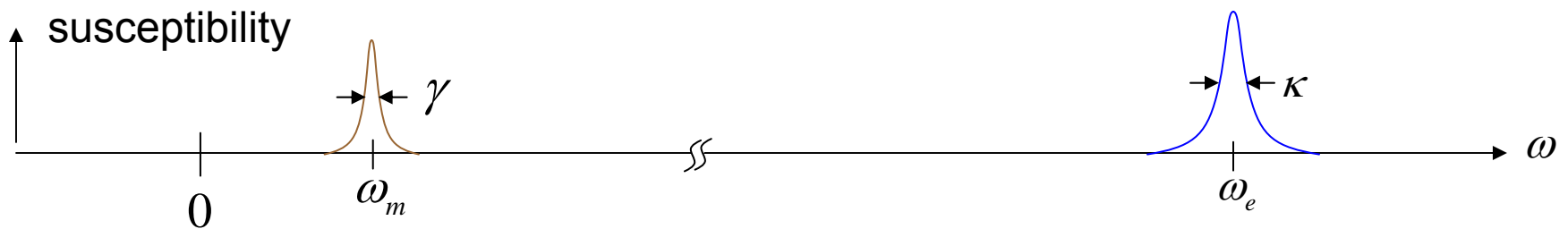
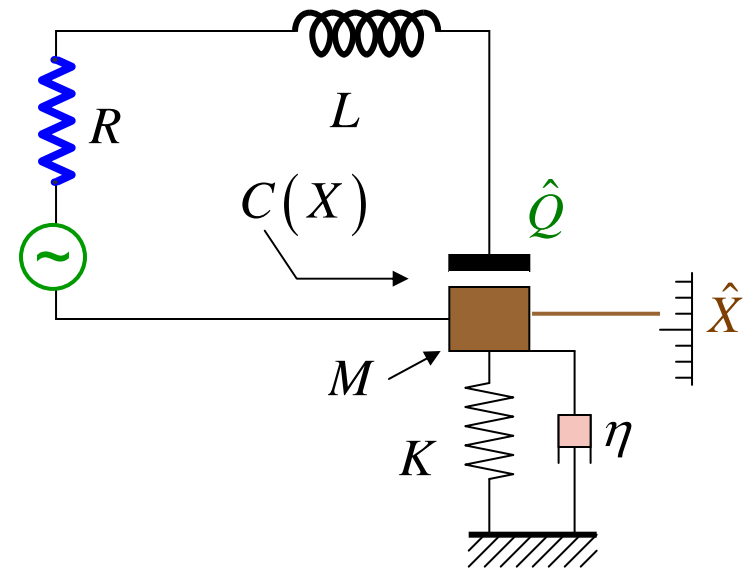
Main references: Marquardt, *et al.*, PRL **99**, 093902 (2007); Marquardt, in "Quantum Machines", Les Houches (2011) *to be published in 2012*; Chan, *et al.*, Nature 478, 89 (2011) and supplement.

COUPLED ELECTRO/OPTO-MECHANICAL OSCILLATORS

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} \left(1 - \frac{\hat{X}}{\ell_0} \right) + \frac{\hat{P}^2}{2M} + \frac{K\hat{X}^2}{2}$$

+ drive + damping

$$\frac{\hat{H}}{\hbar} = \omega_e \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} + g_3 (\hat{b} + \hat{b}^\dagger) \hat{a}^\dagger \hat{a} - \omega_e \alpha (\hat{a} e^{-i\Omega t} + \hat{a}^\dagger e^{+i\Omega t}) + \text{damping}$$

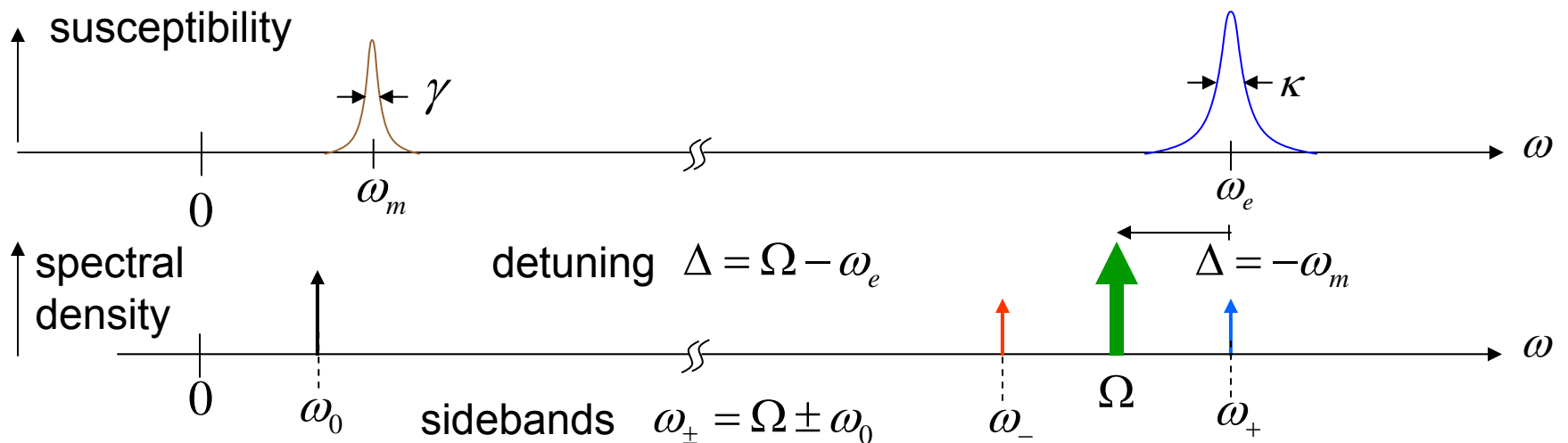
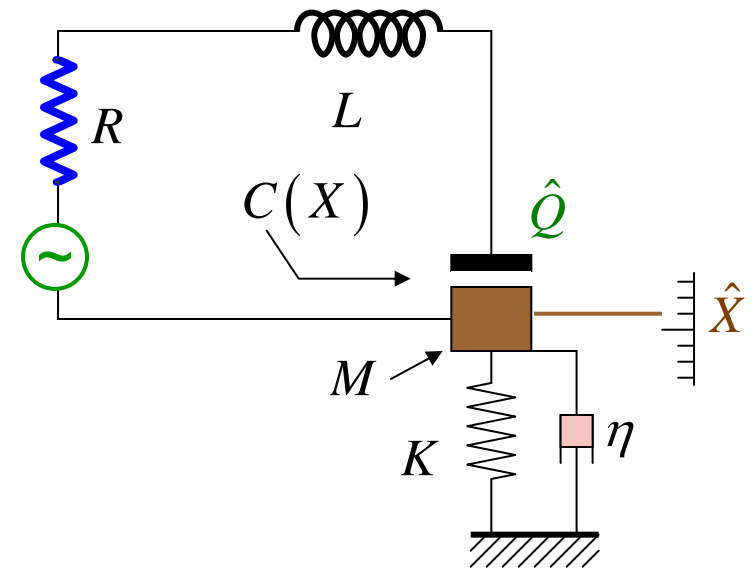


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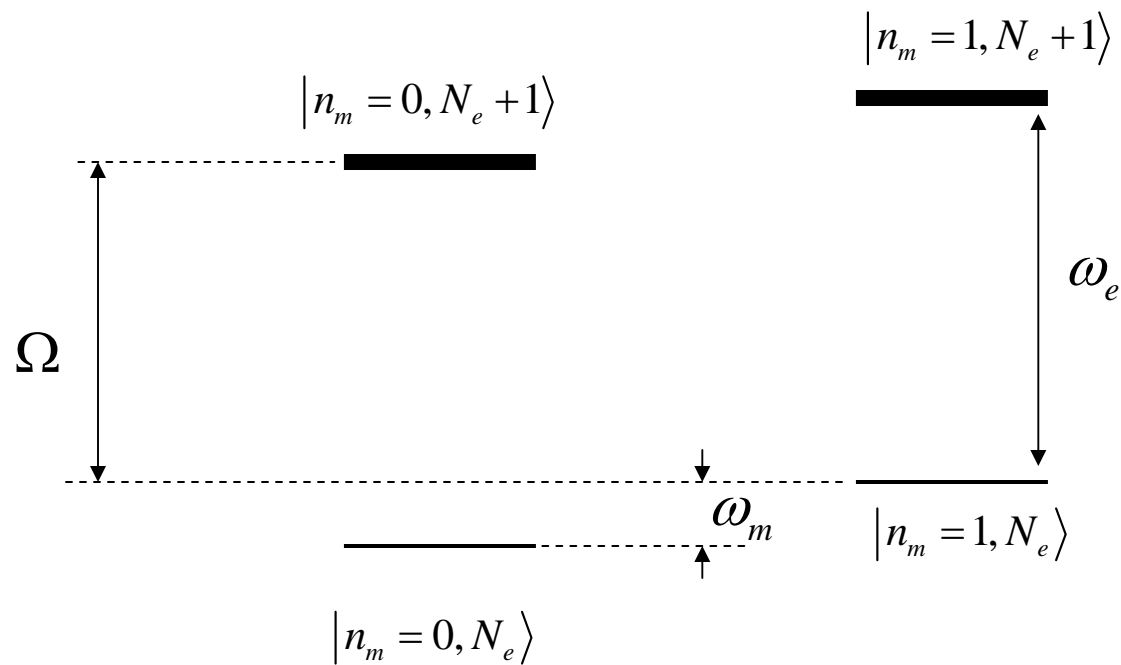
+ drive + damping

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QUALITATIVE DISCUSSION OF COOLING LIMIT (1)

$$\hat{N}_e = \hat{a}^\dagger \hat{a}; \quad \hat{n}_m = \hat{b}^\dagger \hat{b}$$

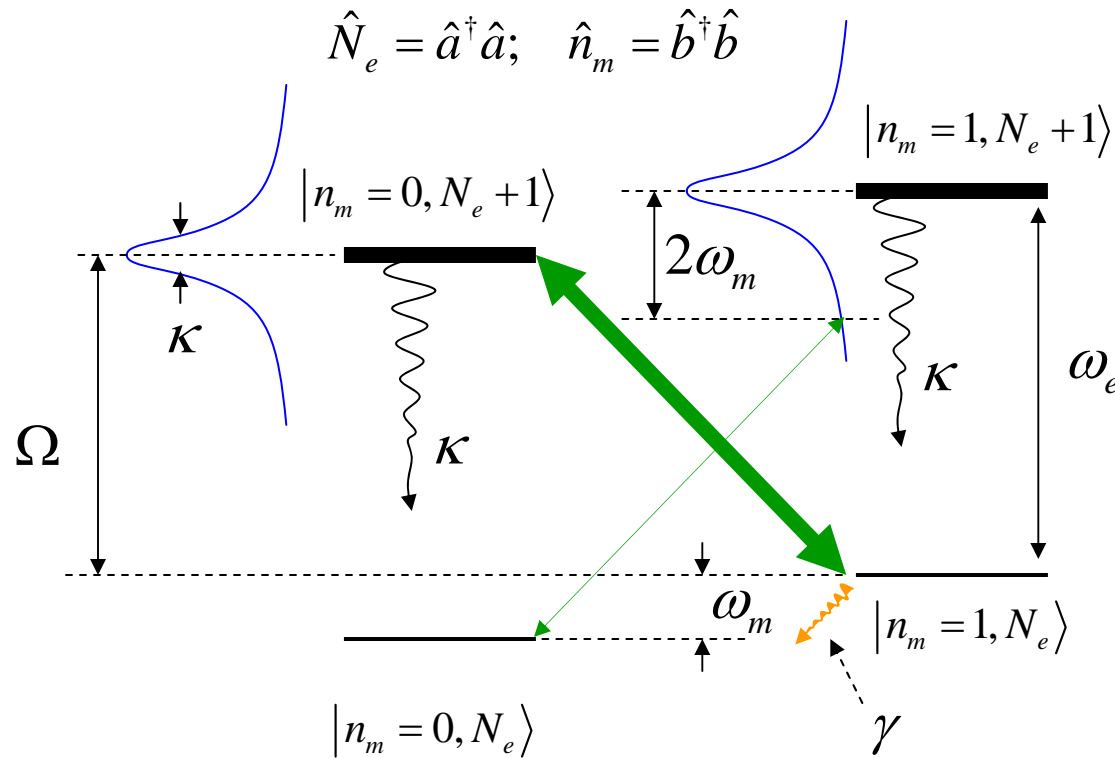


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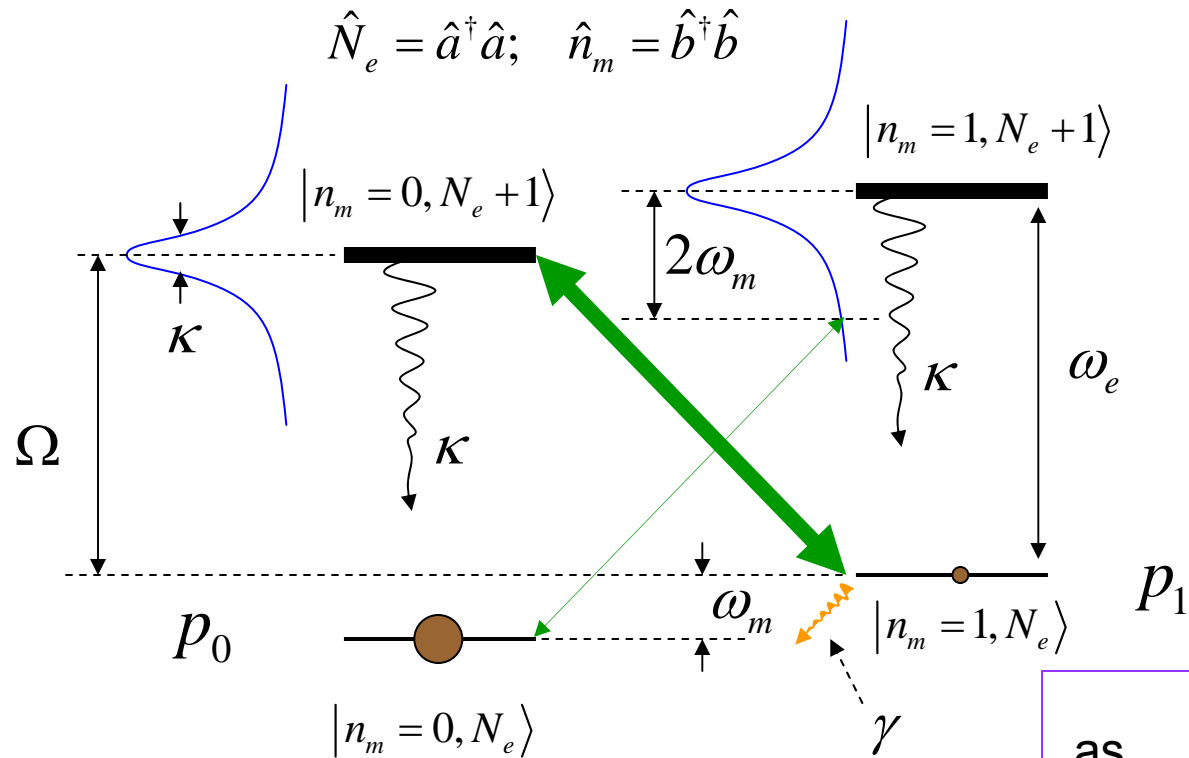
well-resolved sidebands

$$\omega_m \gg \kappa$$



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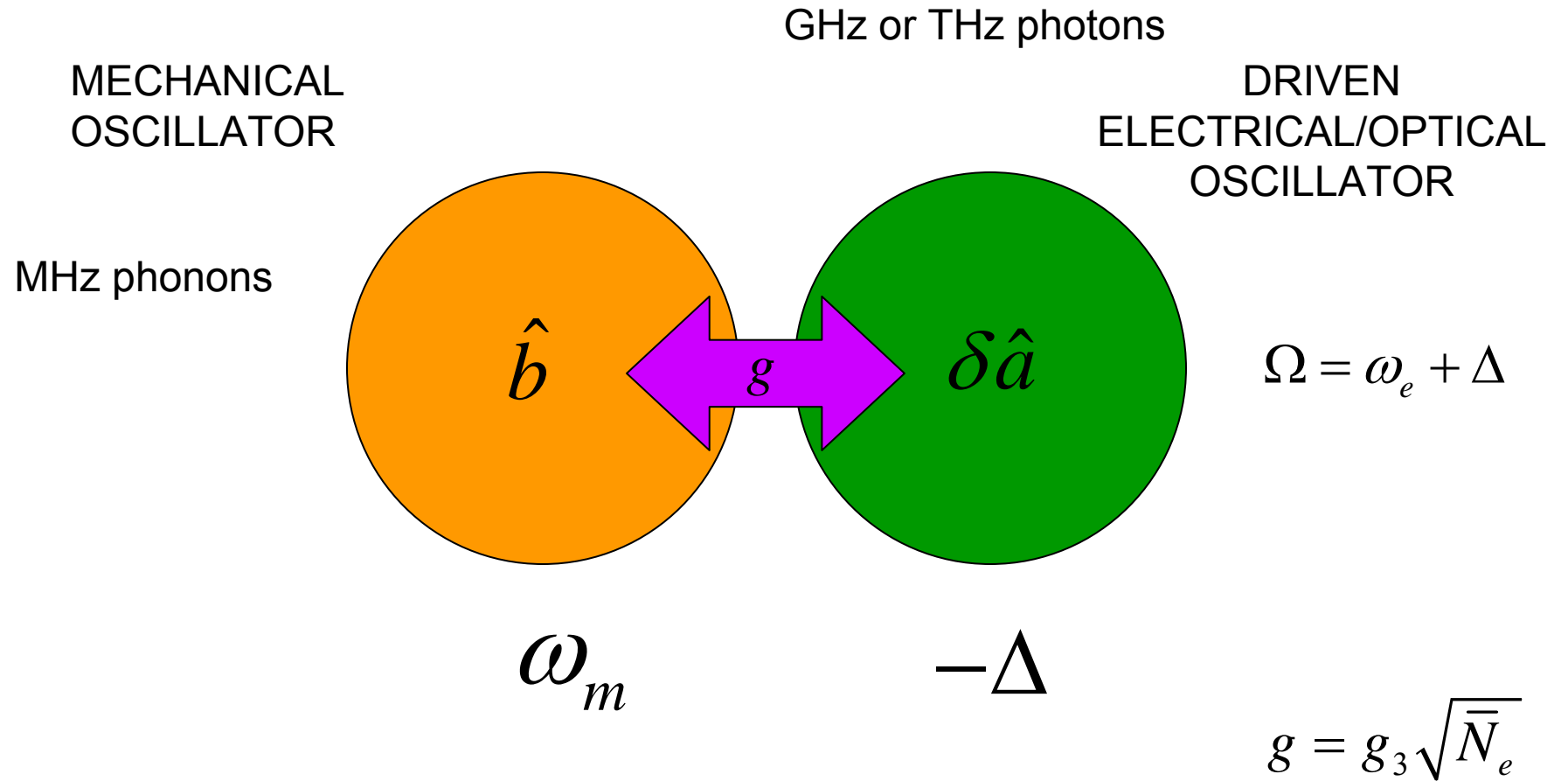
$$\frac{p_1}{p_0} = \exp\left(-\frac{\hbar \omega_m}{k_B T_{cool}^{\min, \gamma \rightarrow 0}}\right)$$

as $\frac{p_1}{p_0} \rightarrow 0 \quad \frac{k_B T_{cool}^{\min, \gamma \rightarrow 0}}{\hbar \omega_m} \rightarrow 0$

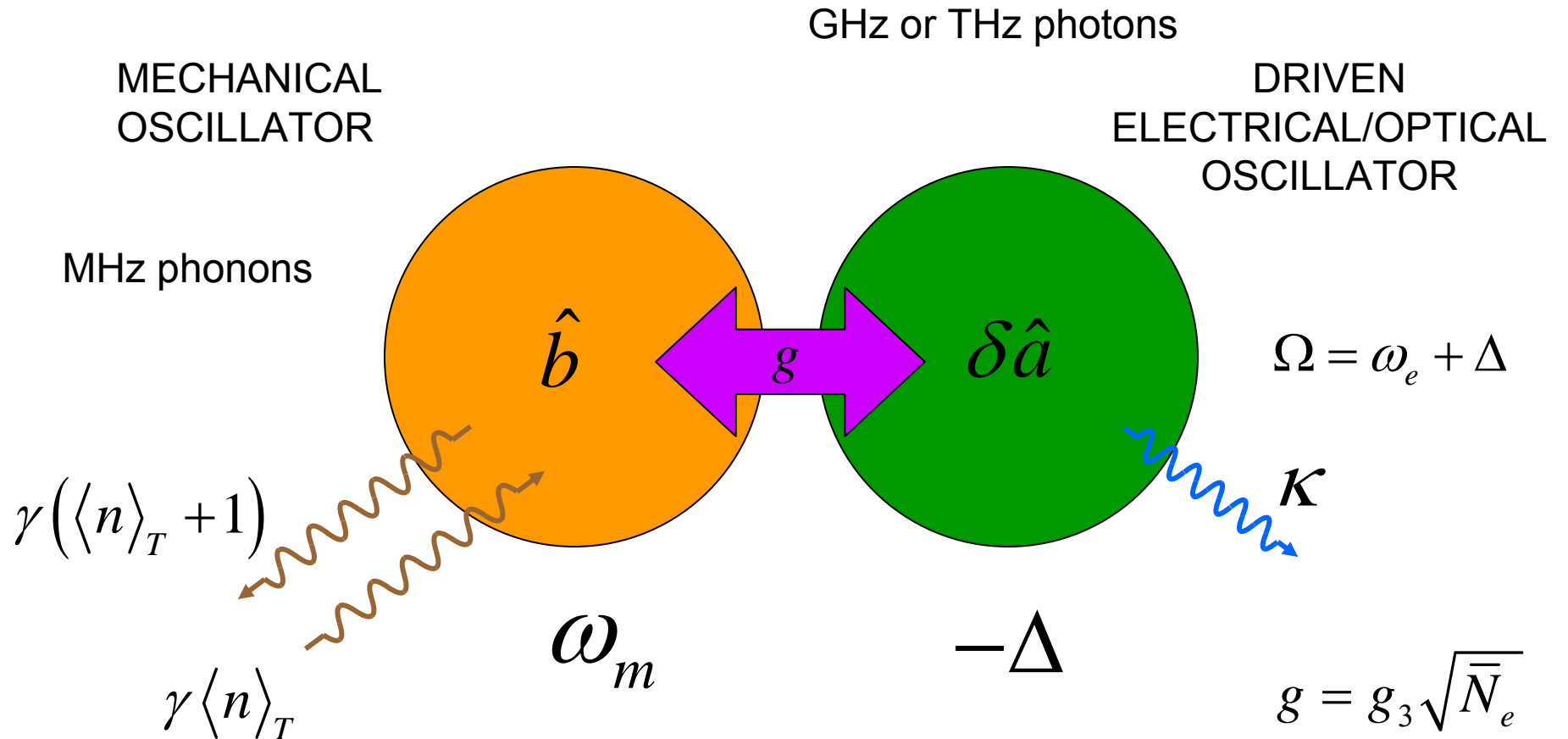
neglect
intrinsic
mechanical
damping
 $\gamma \rightarrow 0$

$$\frac{p_1}{p_0} = \frac{\text{Re}[\chi_{ee}(\omega_e - 2\omega_m)]}{\text{Re}[\chi_{ee}(\omega_e)]} = \frac{(\kappa/2)^2}{(2\omega_m)^2 + (\kappa/2)^2} \rightarrow \left(\frac{\kappa}{4\omega_m}\right)^2$$

QUALITATIVE DISCUSSION OF COOLING LIMIT (2)



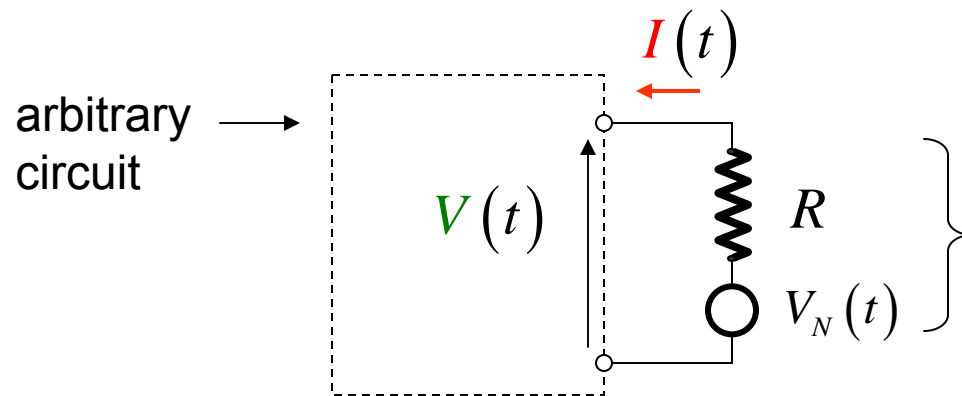
QUALITATIVE DISCUSSION OF COOLING LIMIT (2)



When the mechanical and electromagnetic fluctuation oscillator are hybridized:

$$\frac{dn_m}{dt} \sim -\gamma (n_m - \langle n \rangle_T) - \kappa n_m \quad \Rightarrow \quad n_m^{\min, h} = \frac{\gamma}{\kappa} \langle n \rangle_T$$

NOISE OF A RESISTANCE, FLUCTUATION-DISSIPATION THEOREM



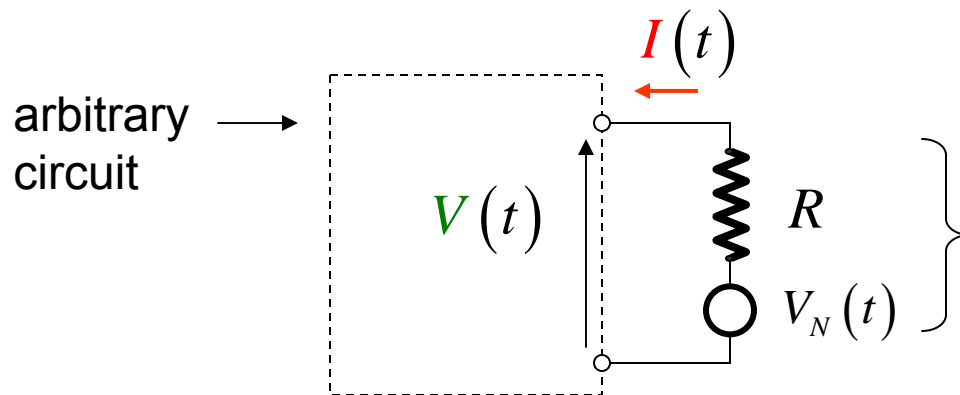
resistance is
always accompanied
by noise voltage source,
whose fluctuations are
uncorrelated with state of
circuit

$$V(t) = -RI(t) + V_N(t)$$

If resistance is in thermal equilibrium at temperature T :

$$\langle V_N(t_1)V_N(t_2) \rangle = 2k_B TR\delta(t_1 - t_2)$$

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In frequency domain:

$$S_{VV}[\omega] = \int_{-\infty}^{+\infty} \langle V_N(t)V_N(0) \rangle e^{i\omega t} dt = 2k_B TR$$

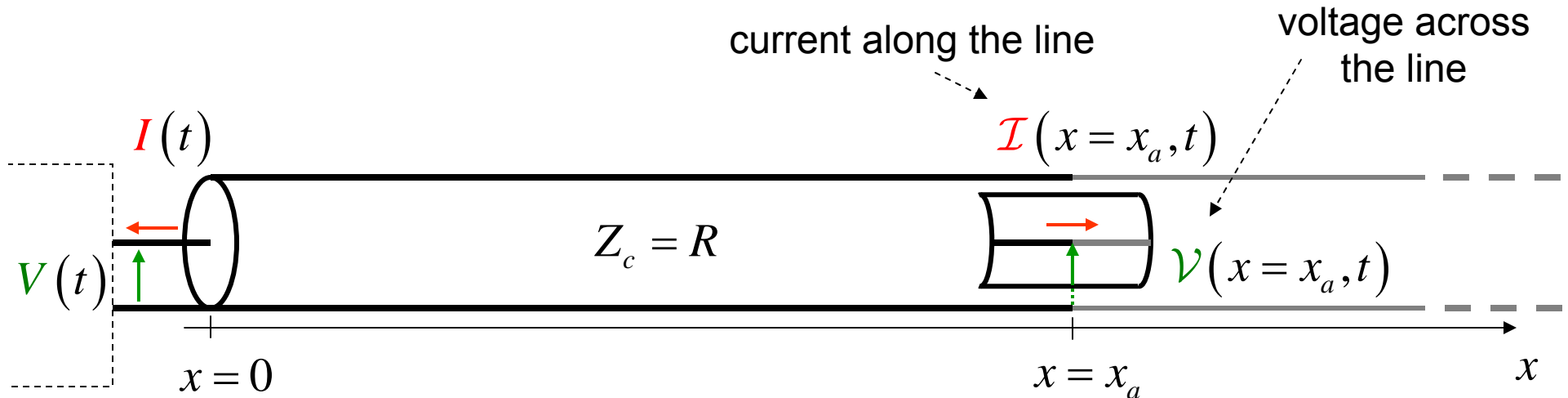
$$S_{VV}(\nu) = S_{VV}[\omega = 2\pi\nu] + S_{VV}[\omega = -2\pi\nu] = 4k_B TR$$

$$V_N[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} V_N(t) e^{i\omega t} dt$$

$$\langle V_N[\omega_1]V_N[\omega_2] \rangle = S_{VV} \left[\frac{\omega_1 - \omega_2}{2} \right] \delta(\omega_1 + \omega_2)$$

QUANTUM-MECHANICALLY, TO DEAL WITH BOTH
DISSIPATION AND FLUCTUATION ASPECTS OF
RESISTANCE, IT IS ADVANTAGEOUS TO REPLACE
IT BY A SEMI-INFINITE TRANSMISSION LINE
(NYQUIST MODEL)

RESISTANCE = SEMI-INFINITE TRANSMISSION LINE

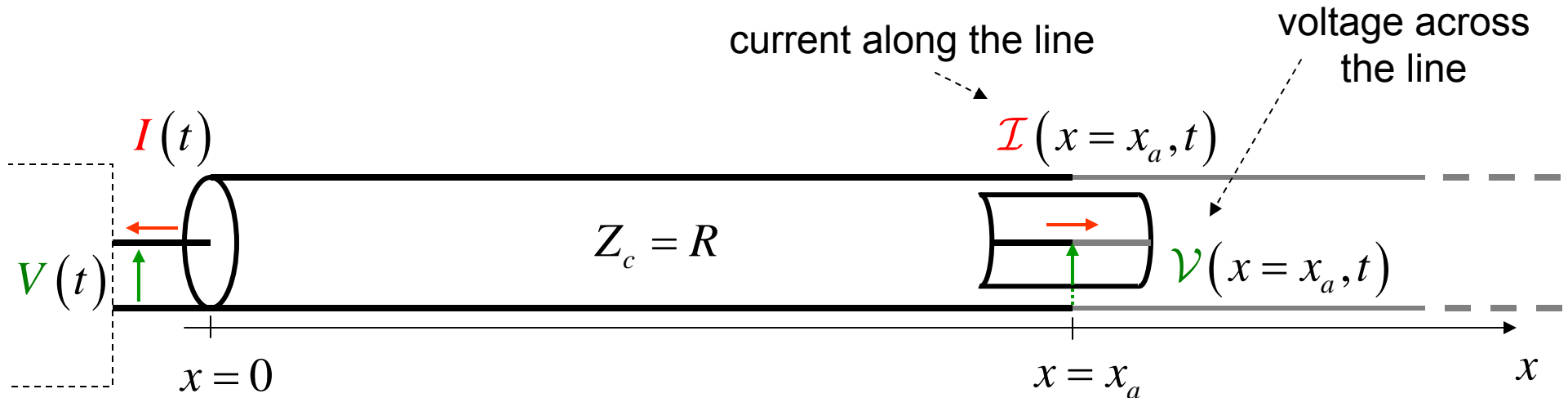


$$I(t) = -\mathcal{I}(x = 0, t)$$

$$V(t) = \mathcal{V}(x = 0, t)$$

$$V(t) = -RI(t) + V_N(t)$$

RESISTANCE = SEMI-INFINITE TRANSMISSION LINE



$$I(t) = -\mathcal{I}(x = 0, t)$$

$$V(t) = \mathcal{V}(x = 0, t)$$

$$V(t) = -RI(t) + V_N(t)$$

$$A^{\text{out}}(x, t) = \frac{1}{2} \left[Z_c^{-1/2} \mathcal{V}(x, t) + Z_c^{+1/2} \mathcal{I}(x, t) \right]$$

$$A^{\text{in}}(x, t) = \frac{1}{2} \left[Z_c^{-1/2} \mathcal{V}(x, t) - Z_c^{+1/2} \mathcal{I}(x, t) \right]$$

$|A^{\text{out,in}}(x, t)|^2$: power flowing in $\pm x$ direction

$$V(t) = -RI(t) + 2\sqrt{R}A^{\text{in}}(0, t)$$

noise = input wave amplitude
at entrance of transmission line,
uncorrelated with present state
of damped circuit

$$A^{\text{out}}(x, t) = A^{\text{out}}(0, t - x/v_p)$$

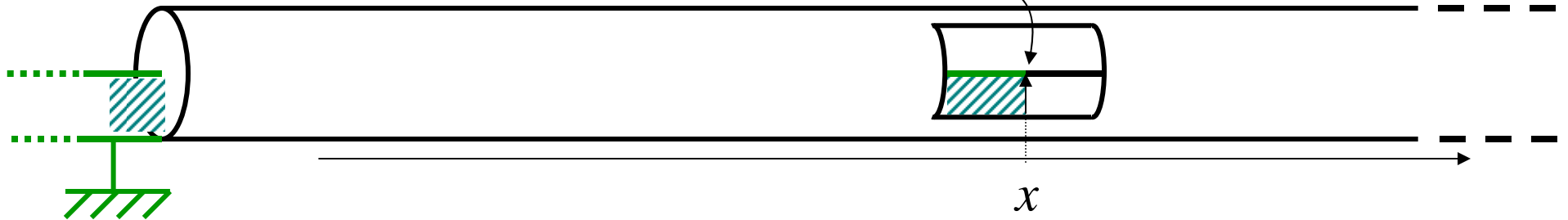
$$A^{\text{in}}(x, t) = A^{\text{in}}(0, t + x/v_p)$$

$$Z_c = \sqrt{\frac{L_\ell}{C_\ell}}$$

$$v_p = \sqrt{\frac{1}{L_\ell C_\ell}}$$

LAGRANGIAN OF TRANSMISSION LINE

For transmission line, field variable is a "node flux": $\Phi(x, t) = \int_{-\infty}^t dt_1 \int_{\text{ground}}^x \vec{E}(x_1, t_1) dx_1$



Lagrangian density:
$$\mathcal{L}(x) = \frac{C_\ell}{2} \left(\frac{\partial \Phi}{\partial t} \right)^2 - \frac{1}{2L_\ell} \left(\frac{\partial \Phi}{\partial x} \right)^2$$

Momentum density:
$$\Pi(x) \equiv \frac{\partial \mathcal{L}}{\partial (\partial \Phi / \partial t)} = C_\ell \frac{\partial \Phi}{\partial t} \quad \underline{\underline{= \text{CHARGE DENSITY ON WIRE}}}$$

Commutation relations:
$$\begin{aligned} [\hat{\Phi}(x_1), \hat{\Pi}(x_2)] &= i\hbar \delta(x_1 - x_2) \\ [\hat{\Phi}(x_1), \hat{\Phi}(x_2)] &= [\hat{\Pi}(x_1), \hat{\Pi}(x_2)] = 0 \end{aligned}$$

COMMUTATION RELATIONS OF WAVE AMPLITUDES

$$A^{\text{in,out}}(x,t) \rightarrow \hat{A}^{\text{in,out}}(x,t)$$

$$\hat{A}^{\text{in,out}}(x,t) = \hat{A}^{\text{in,out}}(0,\tau); \quad \tau = t \pm \frac{x}{v_p}$$

$$\left[\hat{A}^{\text{in,out}}(x_1,t_1), \hat{A}^{\text{in,out}}(x_2,t_2) \right] = \frac{i\hbar}{2} \frac{\partial}{\partial(\tau_1 - \tau_2)} \delta(\tau_1 - \tau_2)$$

$$\hat{A}^{\text{in,out}}(x_1,t_1)^\dagger = \hat{A}^{\text{in,out}}(x_1,t_1)$$

hermitian operator

COMMUTATION RELATIONS OF WAVE AMPLITUDES

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hermitian operator

$$\hat{A}^{\text{in,out}}[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{A}^{\text{in,out}}(x=0, t) e^{i\omega t} dt$$

$$\hat{A}^{\text{in,out}}[\omega]^\dagger = \hat{A}^{\text{in,out}}[-\omega]$$

non-hermitian operator,
with two quadratures
distributed over the pair
of positive and negative
frequencies.

$$\left[\hat{A}^{\text{in,out}}[\omega_1], \hat{A}^{\text{in,out}}[\omega_2] \right] = \frac{\hbar}{2} \left(\frac{\omega_1 - \omega_2}{2} \right) \delta(\omega_1 + \omega_2)$$

WAVE AMPLITUDE vs PHOTON OPERATORS

operator of wave amplitude in Fourier domain

propagation direction

angular frequency (positive or negative)

$$\hat{A}^{in,out}[\omega] = \sqrt{\frac{\hbar |\omega|}{2}} \hat{a}^{in,out}[\omega]$$

$$\left\{ \begin{array}{l} [A^{in,out}[\omega]] = [\text{power}^{1/2} \times \text{time}] = [\text{action}]^{1/2} \\ [a^{in,out}[\omega]] = [\text{time}]^{1/2} \end{array} \right.$$

field ladder operator

$$a^{in,out}[-|\omega|] = a^{in,out}[|\omega|]^\dagger$$

Each frequency $|\omega|$ contains both quadratures of signal

Ladder operators have standard commutation relations described by a single relation:

$$[\hat{a}^{in,out}[\omega_1], \hat{a}^{in,out}[\omega_2]] = \text{sgn}(\omega_1 - \omega_2) \delta(\omega_1 + \omega_2)$$

Scattering always preserves commutation relations.

INPUT NUMBER OF PHOTONS IN FREQUENCY DOMAIN

Input wave amplitude spectral density:

$$\langle A^{in}[\omega_1] A^{in}[\omega_2] \rangle = S_{A^{in}A^{in}}[\omega_1] \delta(\omega_1 + \omega_2) \quad \text{In } T \text{ equilibrium: } S_{A^{in}A^{in}}[\omega] = \frac{\hbar\omega}{4} \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) + 1 \right]$$

$$[S_{AA}[\omega]] = [\text{energy}]$$

$$S_{A^{in}A^{in}}[\omega] \xrightarrow{T \rightarrow \infty} \frac{k_B T}{2}$$

In thermal classical limit, power per unit bandwidth is $k_B T$:

$$\mathcal{S}_{A^{in}A^{in}}(\nu) \xrightarrow{T \rightarrow \infty} k_B T$$

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Photon amplitude spectral density:

$$\langle a[\omega_1] a[\omega_2] \rangle = S_{aa}[\omega_1] \delta(\omega_1 + \omega_2)$$

Thermal spec. dens. $S_{aa}[\omega] = \frac{\text{sgn}(\omega)}{2} \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) + 1 \right]$

Number of photons per unit time per unit bandwidth crossing a section of line:

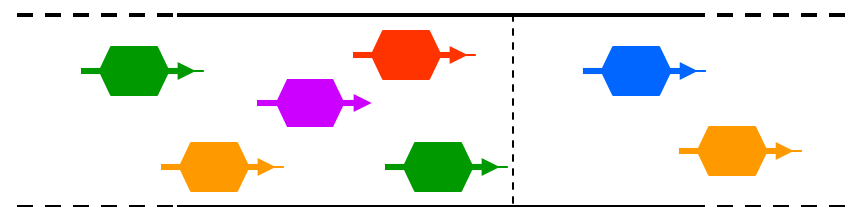
$$N(|\omega|) = S_{aa}[-|\omega|] \quad \leftarrow \text{available photons}$$

$$N_{+ZPF}(|\omega|) = \frac{1}{2} (S_{aa}[+|\omega|] + S_{aa}[-|\omega|]) = N(|\omega|) + \frac{1}{2}$$

Folding of spectral density exhibits quantum noise

Input signal in thermal equilibrium:

$$\left\{ \begin{aligned} N_{+ZPF}^T(|\omega|) &= \frac{1}{2} \coth\left(\frac{\hbar|\omega|}{2k_B T}\right) \\ N_{+ZPF}^T(|\omega|) &\xrightarrow{T \rightarrow \infty} \frac{k_B T}{\hbar|\omega|} \\ N_{+ZPF}^T(|\omega|) &\xrightarrow{T \rightarrow 0} \frac{1}{2} \end{aligned} \right.$$



available photons carry entropy
quantum noise carries no entropy

INPUT PHOTON FIELD IN TIME DOMAIN

Define:

$$\tilde{a}^{in}(t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-i\omega t} \hat{a}^{in}[\omega] d\omega$$

complex operator but
only positive frequencies
contribute

$$\tilde{a}^{in}(t)^\dagger = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{i\omega t} \hat{a}^{in}[-\omega] d\omega$$

complex operator but
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contribute

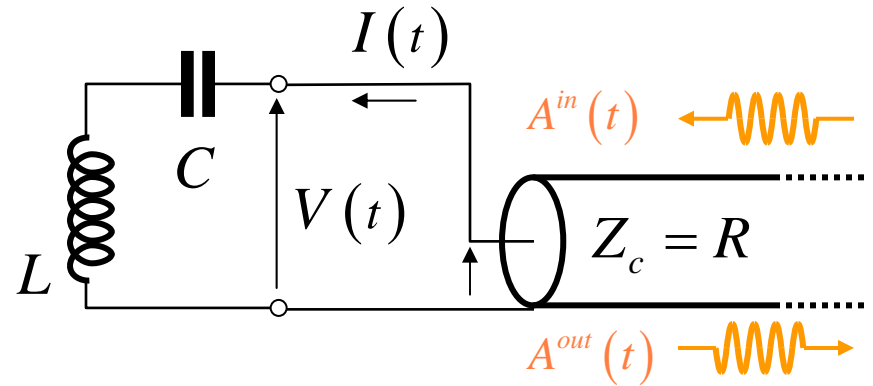
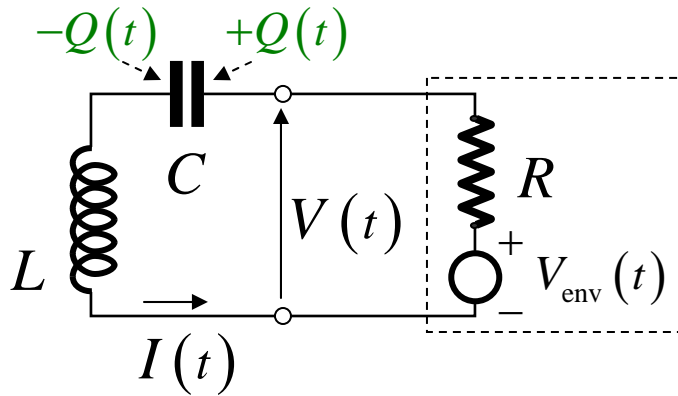
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-i\omega t} \hat{a}^{in}[\omega] d\omega$$

$$\langle \tilde{a}^{in}(t)^\dagger \tilde{a}^{in}(t) \rangle = \langle \dot{N}^{in}(t) \rangle = \frac{1}{2\pi} \int_0^{+\infty} S_{a^{in}a^{in}}[-\omega] d\omega$$

In thermal equilibrium,
with drive at Ω :

$$S_{a^{in}a^{in}}[\omega] = \frac{\text{sgn}(\omega)}{2} \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) + 1 \right] + 2\pi \dot{N}_d \left[\delta(\omega - \Omega) + \delta(\omega + \Omega) \right]$$

DRIVEN, DISSIPATIVE LC SERIES CIRCUIT



$$V_{env}(t) = V_D(t) + V_N(t)$$

deterministic drive signal
noise assoc. w/ dissipation

amplitude of waves on transm. line

$$\begin{cases} A^{in}(t) = \frac{V}{2\sqrt{R}} + \frac{\sqrt{RI}}{2} \\ A^{out}(t) = \frac{V}{2\sqrt{R}} - \frac{\sqrt{RI}}{2} \end{cases}$$

$$V(t) = \frac{Q}{C} + L\ddot{Q} = -R\dot{Q} + V_{env}(t)$$

Kirchhoff's laws \rightarrow equations of motion

input wave carries drive and noise
output wave carries away losses

$$[A^{out}(t)]^2 - [A^{in}(t)]^2 = VI = P(t)$$

$$[A] = [\text{power}]^{1/2}$$

Two point of views:

Standing degree of freedom

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = V_D(t) + V_N(t)$$

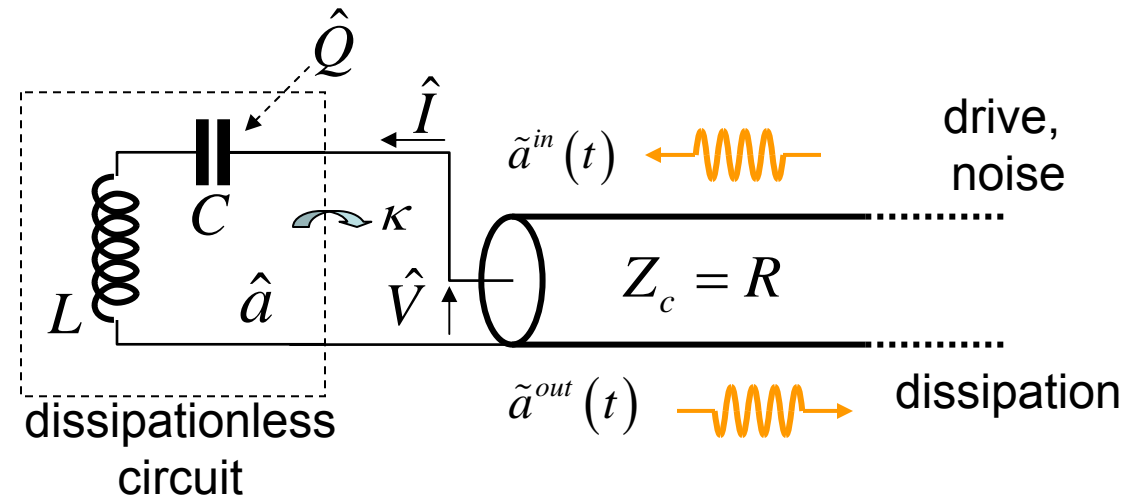
initial conditions: $Q(t=0), \dot{Q}(t=0)$

Scattering of propagating waves

$$\begin{cases} L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = 2\sqrt{R}A^{in}(t) \\ A^{out}(t) = A^{in}(t) - \sqrt{R}\dot{Q} \end{cases}$$

QUANTUM LANGEVIN EQUATION

(Method can be extended if damping is frequency dependent, works if environment has many d. o. freedom)



$$\hat{Q} = Q_{ZPF} (\hat{a} + \hat{a}^\dagger) \quad Q_{ZPF} = \sqrt{\frac{\hbar}{2Z_e}} \quad Z_e = \sqrt{\frac{L}{C}} \quad \omega_e = \sqrt{\frac{1}{LC}}$$

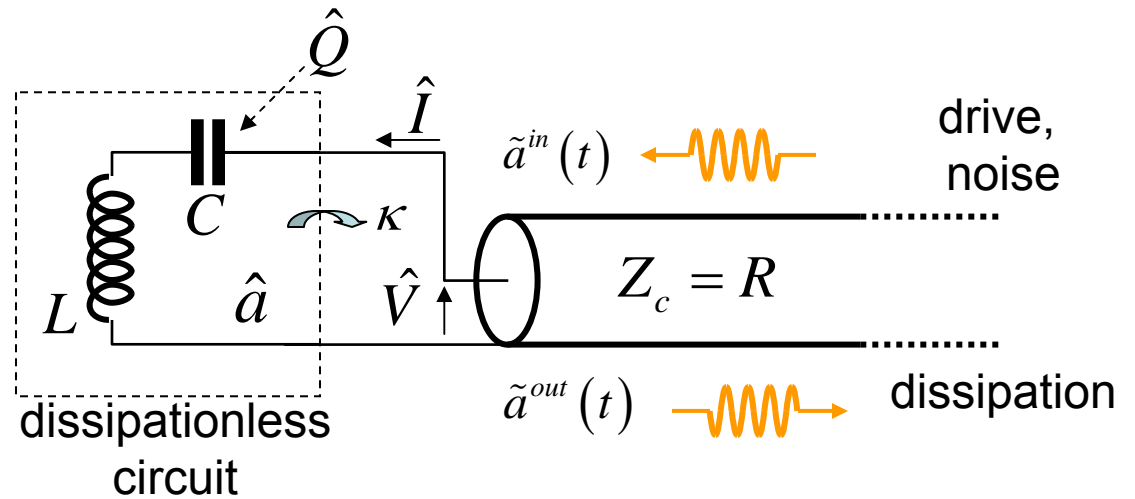
Hamiltonian of isolated dissipationless circuit:

$$\frac{\hat{H}}{\hbar} = \omega_e \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

Without dissipation, \hat{a} has trivial time dependence: $\hat{a}(t) = \hat{a} e^{-i\omega t}$

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$$\hat{Q} = Q_{ZPF} (\hat{a} + \hat{a}^\dagger) \quad Q_{ZPF} = \sqrt{\frac{\hbar}{2Z_e}} \quad Z_e = \sqrt{\frac{L}{C}} \quad \omega_e = \sqrt{\frac{1}{LC}}$$

Hamiltonian of isolated dissipationless circuit:

$$\frac{\hat{H}}{\hbar} = \omega_e \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

Without dissipation: $\hat{a}(t) = \hat{a}e^{-i\omega t}$

$$\text{Rate of photon loss: } \kappa = \omega_e R / Z_e = \frac{R}{L}$$

With dissipation, \hat{a} acquires a non-trivial time dependence:

Quantum Langevin equation:

$$\frac{d}{dt} \hat{a} = \frac{i}{\hbar} [\hat{H}, \hat{a}] - \frac{\kappa}{2} \hat{a} + \sqrt{\kappa} \tilde{a}^{in}$$

$$\tilde{a}^{out} = \tilde{a}^{in} - \sqrt{\kappa} \hat{a}$$

(form given here is valid in under-damped limit)

$$\frac{d}{dt} \hat{a} = -i\omega_e \hat{a} - \frac{\kappa}{2} \hat{a} + \sqrt{\kappa} \tilde{a}^{in}$$

EXERCICE: PHOTON POPULATION OF DAMPED OSCILLATOR IN THERMAL EQUILIBRIUM

Start from Langevin equation:
(valid in very weak damping limit)

$$\frac{d}{dt} \hat{a} = -i\omega_e \hat{a} - \frac{\kappa}{2} \hat{a} + \sqrt{\kappa} \tilde{a}^{in}(t)$$

Go to Fourier domain:

$$-i\omega \hat{a} = -i\omega_e \hat{a} - \frac{\kappa}{2} \hat{a} + \sqrt{\kappa} \tilde{a}^{in}[\omega]$$

Photon amplitude susceptibility:

$$\hat{a}[\omega] = \tilde{\chi}_{aa}[\omega] \tilde{a}^{in}[\omega]$$

$$\tilde{\chi}_{aa}[\omega] = \frac{\sqrt{\kappa}}{\frac{\kappa}{2} - i(\omega - \omega_e)}$$

$$|\tilde{\chi}_{aa}[\omega]|^2 = 2 \frac{\kappa/2}{(\kappa/2)^2 + (\omega - \omega_e)^2}$$

↑ strongly peaked at ω_e ,
integral over pos. freq. = 2π

Photon number in oscillator:

$$\langle N \rangle = \langle a^\dagger a \rangle = \frac{1}{2\pi} \int_0^{+\infty} a[\omega]^\dagger a[\omega] d\omega = \frac{1}{2\pi} \int_0^{+\infty} |\tilde{\chi}_{aa}[\omega]|^2 N^{in}(\omega) d\omega$$

$$\langle N \rangle = \frac{1}{2} \left[\coth\left(\frac{\hbar\omega_e}{2k_B T}\right) - 1 \right] = \left[\exp\left(\frac{\hbar\omega_e}{k_B T}\right) - 1 \right]^{-1}$$

Have recovered stat. mech.
result from scattering treatment!

END OF LECTURE